CSC 311 DATA STRUCTURES Sp '13

Dr. Marek A. Suchenek ©

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1 Some rates of growth of running time $T(n)$ and corresponding rates of growth of max input's size $Max(t)$ and its derivative $Max'(t)$

Let $T(n)$ be a running time of some program P. Let us assert that $T(n)$ is a growing function. This assertion holds in what may be considered a typical situation: the larger the input to program P the longer it takes to process it.

Under the above assertion, $Max(t)$, defined as the maximum size n of input for which $T(n) \leq t$, is the inverse of $T(n)$, that is,

$$t = T(n) \text{ iff } n = Max(t). \tag{1}$$

In particular, $Max(t)$ is a growing function as well. (Proof left as an easy exercise for the reader.)

The following fact holds for every differentiable growing function f:

$$(f_{inverse})'(x) = \frac{1}{f'(f_{inverse}(x)),} \tag{2}$$

where $f_{inverse}$ is the inverse of f (it exists since f is a growing function) and f' is the derivative of f. (Proof left as an intermediate exercise for the reader - requires calculus. *Hint*: Use the geometric/trigonometric interpretation of derivative.)

In particular,

$$Max'(t) = \frac{1}{T'(Max(t)).} \tag{3}$$

The derivative $Max'(t)$ of $Max(t)$ seems like a good measure of return on investment of a faster computer (or - equivalently - longer wait) for program P. It tells how fast (or slow) the maximum size of tractable input to P will grow with the increase of the computer's speed. So, the larger $Max'(t)$ the more cost effective it is at point t to run P on a faster computer. And vice versa: the smaller $Max'(t)$ the more wasteful it is at point t to run P on a faster computer.

When the measure $Max'(t)$ is decreasing then it might be insightful to consider also the reciprocal $\frac{1}{Max'(t)}$ of $Max'(t)$ as a measure of the cost of enlargement of maximum size $Max(t)$ of tractable input to program P. In such a case, it seems easier to evaluate visually the rate of growth of $\frac{1}{Max'(t)}$ than the rate of decline of $Max'(t)$ based on their graphs. The measure $\frac{1}{Max'(t)}$ tells how much faster (or longer) the program P must be executed in order to accomplish the unit increase of tractable input to P. So, the larger the $\frac{1}{Max'(t)}$ the more costly it is at point t to run P on even a slightly larger input.

By (3),

$$\frac{1}{Max'(t)} = T'(Max(t)). \tag{4}$$

Finding the Θ characterization of $f'(t)$ and of $\frac{1}{f'(t)}$ from the Θ characterization of $f(t)$ requires some extra assumption that f and its Θ benchmark (representative) satisfy assumptions of the de l'Hôpital rule.

Theorem 1.1 *Let f and g be positive, increasing, differentiable functions that both converge to 0 or both diverge to ∞ as their arguments diverge to ∞. Assume that $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists. Then*

$$f \in \Theta(g) \equiv f' \in \Theta(g') \equiv \frac{1}{f'} \in \Theta(\frac{1}{g'}).$$

Proof (optional for all students). $f' \in \Theta(g')$ iff [because $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists] $0 < \lim_{x\to\infty} \frac{f'(x)}{g'(x)} < \infty$ iff (by de l'Hôpital rule) $0 < \lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$ iff [because $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists] iff $f \in \Theta(g)$. This completes the proof of the first equivalence.

$f' \in \Theta(g')$ iff [because $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists] $0 < \lim_{x\to\infty} \frac{f'(x)}{g'(x)} < \infty$ iff $0 < \lim_{x\to\infty} \frac{g'(x)}{f'(x)} < \infty$ iff $0 < \lim_{x\to\infty} \frac{\frac{1}{f'(x)}}{\frac{1}{g'(x)}} < \infty$ iff [because $\lim_{x\to\infty} \frac{\frac{1}{f'(x)}}{\frac{1}{g'(x)}} < \infty$ exists] $\frac{1}{f'} \in \Theta(\frac{1}{g'})$. This completes the proof of the second equivalence. This completes the proof. □

Below, several examples of $T(n)$ and corresponding $Max(t)$ and the derivative $Max'(t)$ are described. For the cases 8 through 13 the reciprocals $\frac{1}{Max'(t)}$ are included. All cases 1 through 13 may be considered benchmark cases.

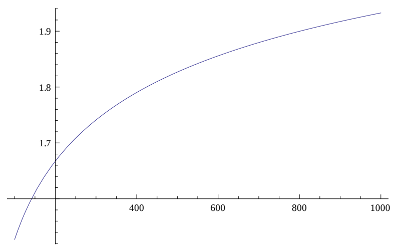
Particularly important are cases: 1, 3, 7, 8, 9, 11, and 12.

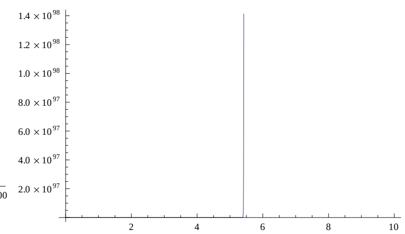
Note different scales used in graphs of sample functions below.

1.

$$T(n) \in \Theta(\log \log n) \ldots Max(t) \in \Theta(a^{b^t}); \text{ for some } a, b > 1$$

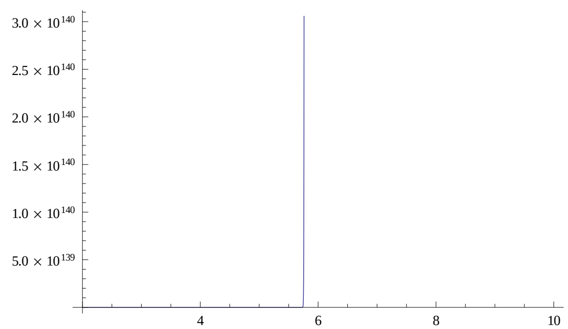
Here are graphs of $\ln \ln n$ and e^{e^t}:





$$Max'(t) \in \Theta(a^{b^t} b^t)$$

Here is a graph of $e^{e^t} e^t = e^{e^t + t}$:

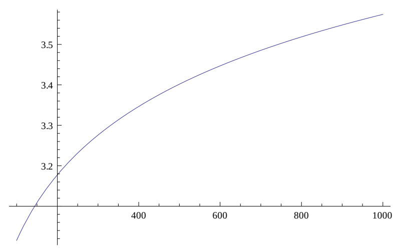


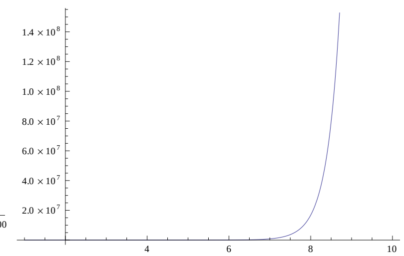
In this case, the larger t the (dramatically) more it pays off to run P on a faster computer.

2.

$$T(n) \in \Theta(\frac{\log n}{\log\log n}) \ldots Max(t) \in \Omega(at)^{at}) \cap O((bt)^{bt}); \text{ for some } a, b > 1$$

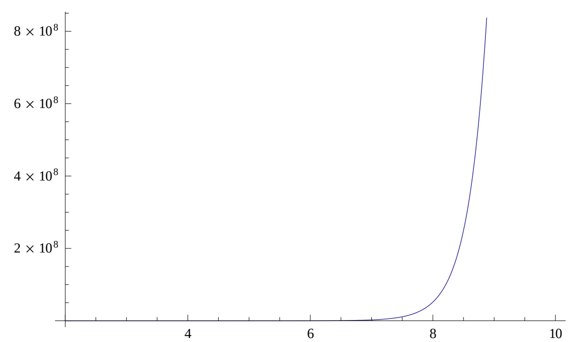
Here are graphs of $\frac{\log n}{\log\log n}$ and t^t:





$$Max'(t) \in \Omega((at)^{at} \ln t) \cap O((bt)^{bt} \ln t)$$

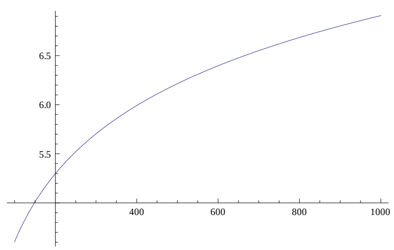
Here is a graph of $t^t \ln t$:

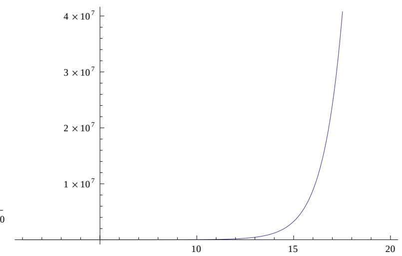


In this case, the larger t the (significantly) more it pays off to run P on a faster computer.

3.
$$T(n) \in \Theta(\log n) \ldots Max(t) \in \Theta(a^t); \text{ for some } a > 1$$

Here are graphs of $\ln n$ and e^t:





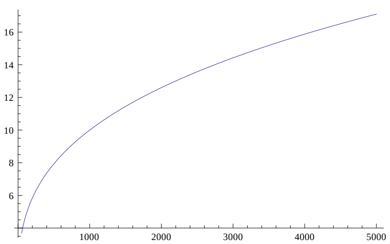
$$Max'(t) \in \Theta(a^t)$$

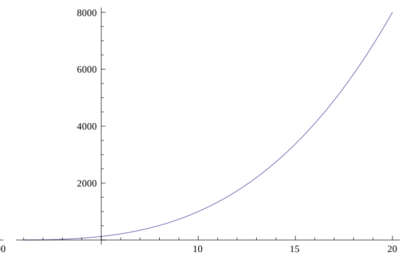
See above for a graph of e^t.

In this case, the larger t the (significantly) more it pays off to run P on a faster computer.

4.
$$T(n) \in \Theta(\sqrt[3]{n}) \ldots Max(t) \in \Theta(t^3)$$

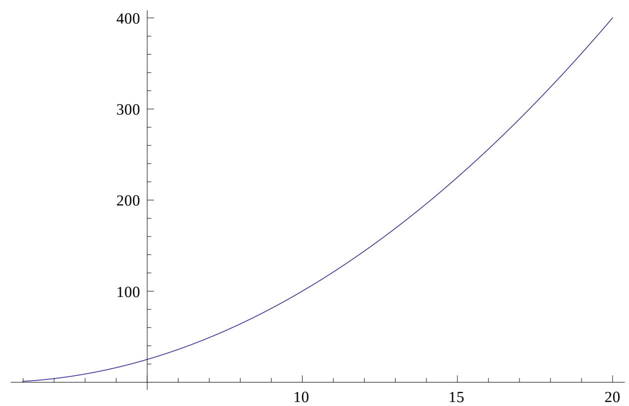
Here are graphs of $\sqrt[3]{n}$ and t^3:





$$Max'(t) \in \Theta(t^2)$$

Here is a graph of t^2:

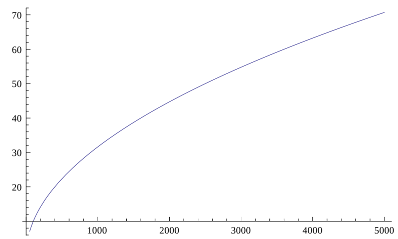


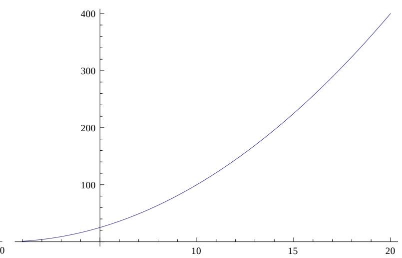
In this case, the larger t the more it pays off to run P on a faster computer.

5.

$$T(n) \in \Theta(\sqrt{n}) \ldots Max(t) \in \Theta(t^2)$$

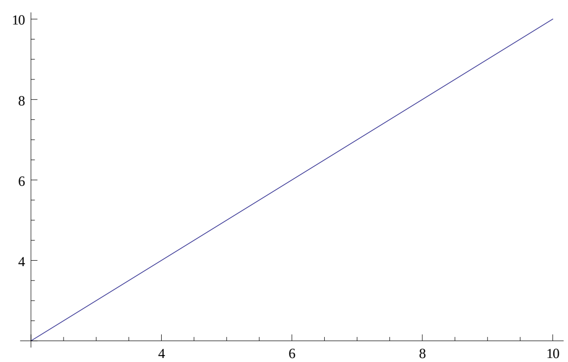
Here are graphs of $\sqrt{n}$ and t^2:





$$Max'(t) \in \Theta(t)$$

Here is a graph of t:

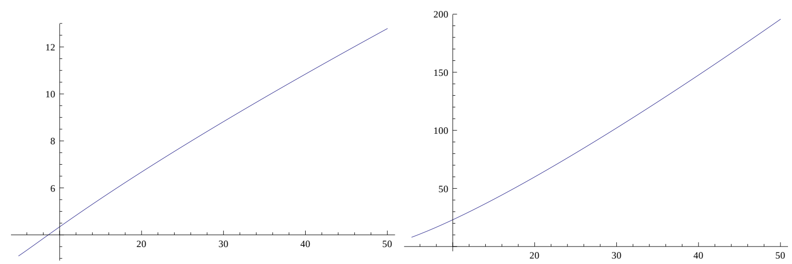


In this case, the larger t the more it pays off to run P on a faster computer.

6.

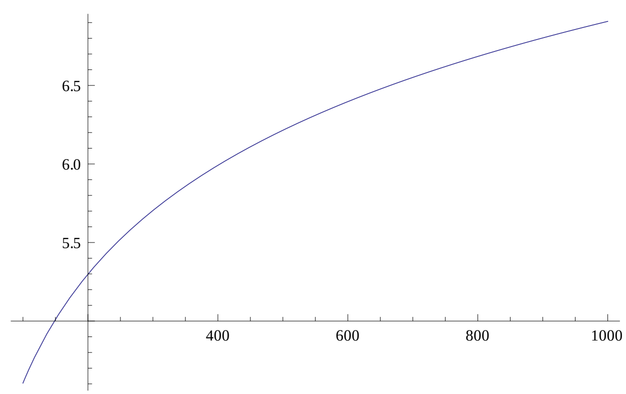
$$T(n) \in \Theta(\frac{n}{\log n}) \ldots Max(t) \in \Theta(t \log t)$$

Here are graphs of of $\frac{n}{\log n}$ and $t \ln t$:



$$Max'(t) \in \Theta(\log t)$$

Here is a graph of $\ln n$:

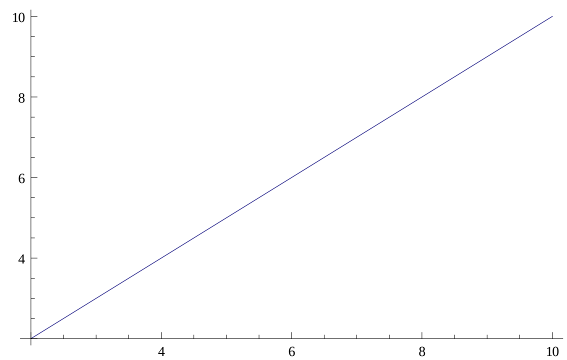


In this case, the larger t the (moderately) more it pays off to run P on a faster computer.

7.

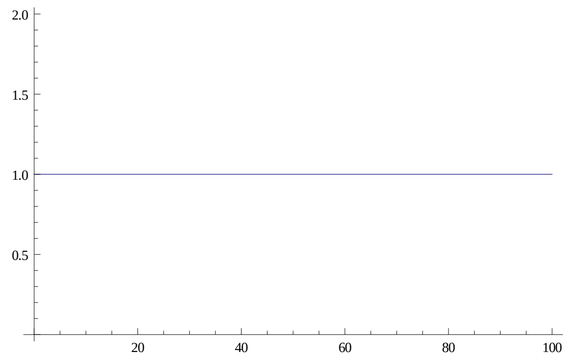
$$T(n) \in \Theta(n) \ldots Max(t) \in \Theta(t)$$

Here is a graphs of n and t:



$$Max'(t) \in \Theta(1)$$

Here is a graph of 1:

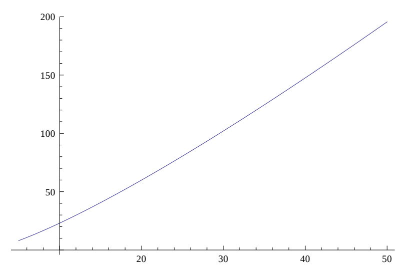


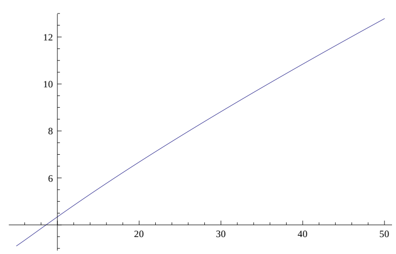
In this case, the increase of the maximum size of input in function of speed of the computer is constant for all t, so the payoff for running P on a faster computer remains roughly the same for all sizes of its inputs.

8.

$$T(n) \in \Theta(n \log n) \ldots Max(t) \in \Theta(\frac{t}{\log t})$$

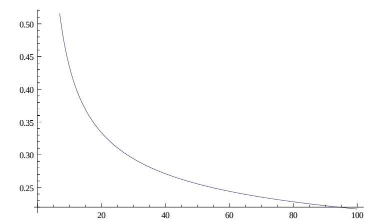
Here are graphs of $n \log n$ and $\frac{t}{\log t}$:

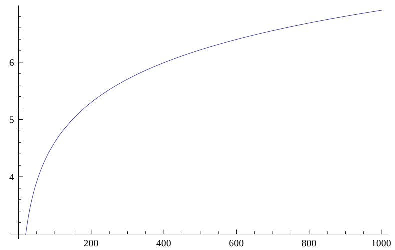




$$Max'(t) \in \Theta(\frac{1}{\log t});\ \frac{1}{Max'(t)} \in \Theta(\log t)$$

Here are graphs of $\frac{1}{\log t}$ and $\log t$:



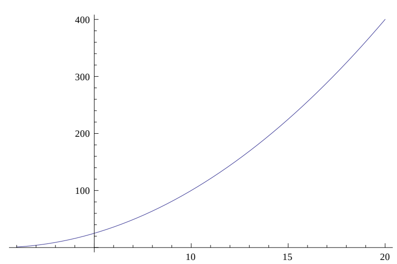


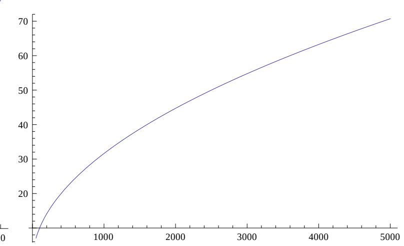
In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (slightly) more does it cost to accomplish the unit increase of the tractable input to P.

9.

$$T(n) \in \Theta(n^2) \ldots Max(t) \in \Theta(\sqrt{t})$$

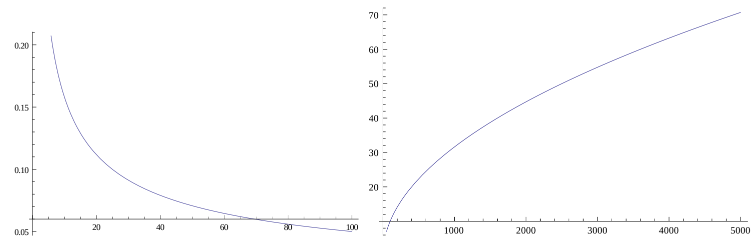
Here are graphs of n^2 and $\sqrt{t}$:





$$Max'(t) \in \Theta(\frac{1}{\sqrt{t}});\ \frac{1}{Max'(t)} \in \Theta(\sqrt{t})$$

Here are graphs of $\frac{1}{\sqrt{t}}$ and $\sqrt{t}$:

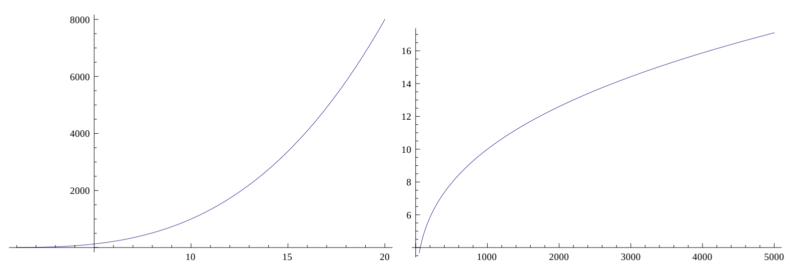


In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (moderately) more does it cost to accomplish the unit increase of the tractable input to P.

10.

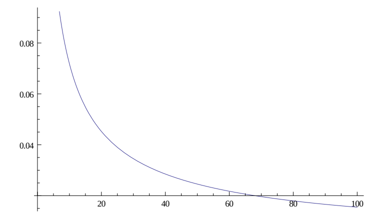
$$T(n) \in \Theta(n^3) \ldots Max(t) \in \Theta(\sqrt[3]{t})$$

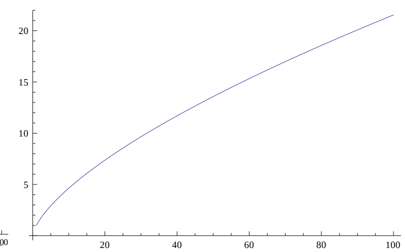
Here are graphs of n^3 and $\sqrt[3]{t}$:



$$Max'(t) \in \Theta(\frac{1}{\sqrt[3]{t^2}});\ \frac{1}{Max'(t)} \in \Theta(\sqrt[3]{t^2})$$

Here are graphs of $\frac{1}{\sqrt[3]{t^2}}$ and $\sqrt[3]{t^2}$:



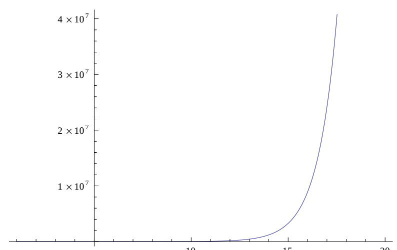


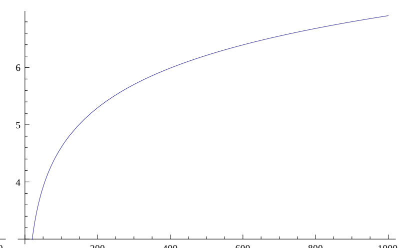
In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (moderately) more does it cost to accomplish the unit increase of the tractable input to P.

11.

$$T(n) \in \Theta(a^n) \ldots Max(t) \in \Theta(\log t);\ \text{for all } a > 1$$

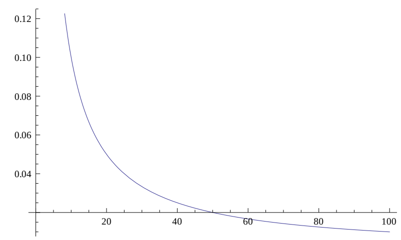
Here are graphs of e^n and $\ln t$:

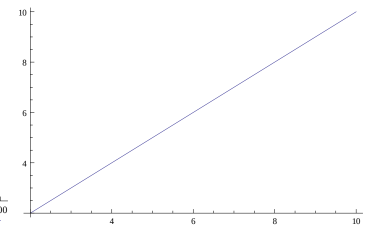




$$Max'(t) \in \Theta(\frac{1}{t}); \frac{1}{Max'(t)} \in \Theta(t)$$

Here are graphs of $\frac{1}{t}$ and t:



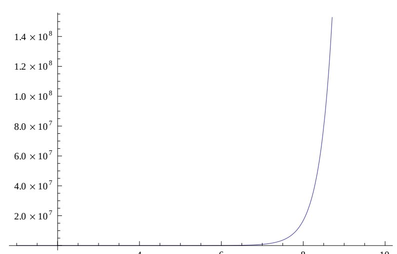


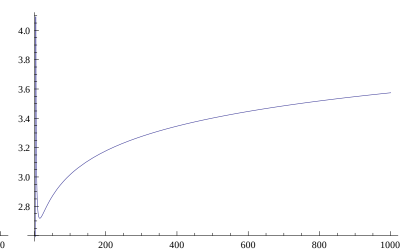
In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P.

12.

$$T(n) \in \Theta((an)^{bn}) \ldots Max(t) \in \Theta(\frac{\log t}{\log \log t}); \text{ for all } a, b > 1$$

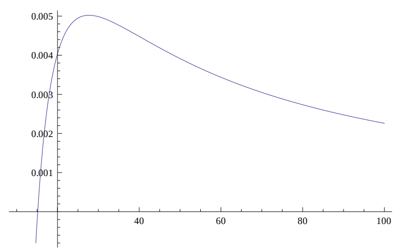
Here are graphs of n^n and $\frac{\log t}{\log \log t}$:

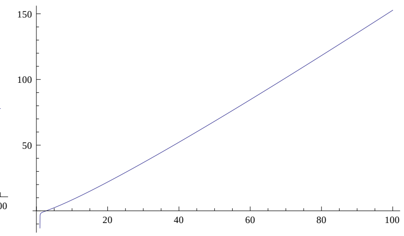




$$Max'(t) \in \Theta(\frac{1}{t \ln \ln t} - \frac{1}{t(\ln \ln t)^2}) = \Theta(\frac{1}{t \ln \ln t}); \; \frac{1}{Max'(t)} \in \Theta(t \ln \ln t)$$

Here are graphs of $\frac{1}{t \ln \ln t} - \frac{1}{t(\ln \ln t)^2}$ and $t \ln \ln t$:



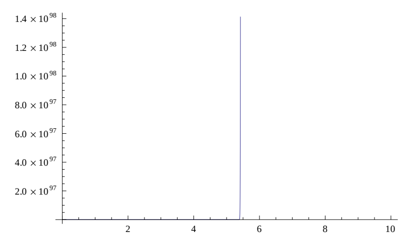


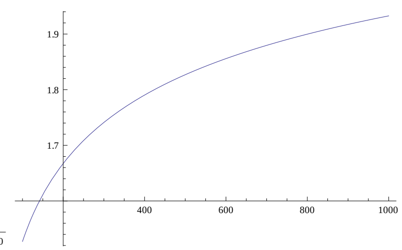
In this case, the larger t the (dramatically) less it pays off to run P on a faster computer. More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P.

13.

$$T(n) \in \Theta(a^{b^n}) \ldots Max(t) \in \Theta(\log \log t); \text{ for all } a, b > 1$$

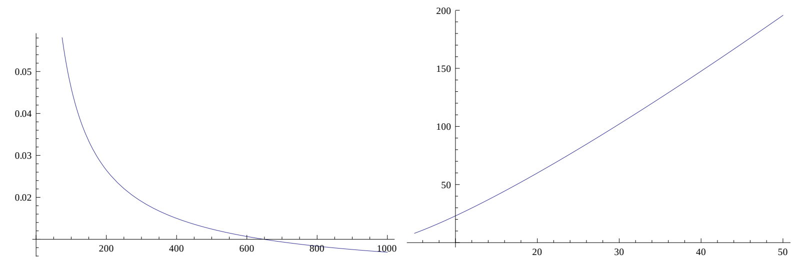
Here are graphs of e^{e^n} and $\log \log t$:





$$Max'(t) \in \Theta(\frac{1}{t \ln t}); \ \frac{1}{Max'(t)} \in \Theta(t \ln t)$$

Here are graphs of $\frac{1}{t \ln t}$ and $t \ln t$:



In this case, the larger t the (dramatically) less it pays off to run P on a faster computer.More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P.

1.1 Proofs (optional for all students)

1. The inverse function of the function

$$n = a^{b^t}$$

is

$$t = \log_b \log_a n.$$

Since for every $a, b > 0$,

$$\lim_{n\to\infty} \frac{\log_b \log_a n}{\ln \ln n}$$

exists and is between 0 and ∞,

$$\Theta(\log_b \log_a n) = \Theta(\log \log n).$$

2. The inverse function of the function

$$n = (at)^{at}$$

is

$$t = \frac{\ln n}{aW(\ln n)}$$

where $W(x)$ is the Lambert's W function approximated by:

$$W(x) = \ln x - \ln \ln x \pm O(\frac{\ln \ln x}{\ln x}).$$

Therefore, the inverse function of the function

$$n = (at)^{at}$$

is in

$$\Theta(\frac{\ln n}{\ln \ln n - \ln \ln \ln n}).$$

Since

$$\lim_{n\to\infty} \frac{\frac{\ln n}{\ln\ln n - \ln\ln\ln n}}{\frac{\ln n}{\ln\ln n}}$$

exists and is between 0 and ∞,

$$\Theta(\frac{\ln n}{\ln\ln n - \ln\ln\ln n}) = \Theta(\frac{\ln n}{\ln\ln n}).$$

For $0 < a < b$ we have $(at)^{at} < (bt)^{at} < (at)^{bt} < (bt)^{bt}$. Hence, for each increasing function $f \in \Theta(\frac{\ln n}{\ln\ln n})$, and some $a > 0$, $f^{-1} \in \Omega((at)^{at})$. Similarily, for some $b > 0$, $f^{-1} \in O((bt)^{bt})$.

3. The inverse function of the function

$$n = a^t$$

is

$$t = \log_a n.$$

Since for every $a > 0$,

$$\lim_{n\to\infty} \frac{\log_a n}{\ln n}$$

exists (it's $\frac{1}{\ln a}$) and is between 0 and ∞,

$$\Theta(\log_a n) = \Theta(\log n).$$

4. Left as an exercise.

5. Left as an exercise.
6. The inverse function of the function

$$n = t \log_a t$$

is

$$t = \frac{n \ln a \ln n}{W(n \ln a \ln n)}$$

where $W(x)$ is the Lambert's W function approximated by:

$$W(x) = \ln x - \ln \ln x \pm O(\frac{\ln \ln x}{\ln x}).$$

Therefore, the inverse function of the function

$$t = n \log_a n$$

is in

$$\Theta(\frac{n \ln a \ln n}{\ln(n \ln a \ln n) - \ln \ln(n \ln a \ln n)}).$$

Since

$$\lim_{n \to \infty} \frac{\frac{n \ln a \ln n}{\ln(n \ln a \ln n) - \ln \ln(n \ln a \ln n)}}{\frac{n}{\ln n}}$$

exists and is between 0 and ∞,

$$\Theta(\frac{n \ln a \ln n}{\ln(n \ln a \ln n) - \ln\ln(n \ln a \ln n)}) = \Theta(\frac{n}{\ln n}).$$

7. Left as an exercise.

8. The inverse function of the function

$$t = n \log_a n$$

is

$$n = \frac{t \ln a \ln t}{W(t \ln a \ln t)}$$

where $W(x)$ is the Lambert's W function approximated by:

$$W(x) = \ln x - \ln\ln x \pm O(\frac{\ln\ln x}{\ln x}).$$

Therefore, the inverse function of the function

$$n = t \log_a t$$

is in

$$\Theta(\frac{t \ln a \ln t}{\ln(t \ln a \ln t) - \ln\ln(t \ln a \ln t)}).$$

Since

$$\lim_{t\to\infty} \frac{\frac{t\ln a\ln t}{\ln(t\ln a\ln t)-\ln\ln(t\ln a\ln t)}}{\frac{t}{\ln t}}$$

exists and is between 0 and ∞,

$$\Theta(\frac{t\ln a\ln t}{\ln(t\ln a\ln t)-\ln\ln(t\ln a\ln t)}) = \Theta(\frac{t}{\ln t}).$$

9. Left as an exercise.

10. Left as an exercise.

11. The inverse function of the function

$$t = a^n$$

is

$$n = \log_a t.$$

Since for every $a > 0$,

$$\lim_{t\to\infty} \frac{\log_a t}{\ln t}$$

exists (it's $\frac{1}{\ln a}$) and is between 0 and ∞,

$$\Theta(\log_a t) = \Theta(\log t).$$

12. The inverse function of the function

$$t = (an)^{an}$$

is

$$n = \frac{\ln t}{aW(\ln t)}$$

where $W(x)$ is the Lambert's W function approximated by:

$$W(x) = \ln x - \ln\ln x \pm O(\frac{\ln\ln x}{\ln x}).$$

Therefore, the inverse function of the function

$$t = (an)^{an}$$

is in

$$\Theta(\frac{\ln t}{\ln\ln t - \ln\ln\ln t}).$$

Since

$$\lim_{t\to\infty} \frac{\frac{\ln t}{\ln\ln t - \ln\ln\ln t}}{\frac{\ln t}{\ln\ln t}}$$

exists and is between 0 and ∞,

$$\Theta(\frac{\ln t}{\ln\ln t - \ln\ln\ln t}) = \Theta(\frac{\ln t}{\ln\ln t}).$$

For $0 < a < b$ we have $(an)^{an} < (bn)^{an} < (an)^{bn} < (bn)^{bn}$. Hence, for each increasing function $f \in \Theta(\frac{\ln n}{\ln \ln n})$, and some $a > 0$, $f^{-1} \in \Omega((an)^{an})$. Similarily, for some $b > 0$, $f^{-1} \in O((bn)^{bn})$.

13. The inverse function of the function

$$t = a^{b^n}$$

is

$$n = \log_b \log_a t.$$

Since for every $a, b > 0$,

$$\lim_{t \to \infty} \frac{\log_b \log_a t}{\ln \ln t}$$

exists and is between 0 and ∞,

$$\Theta(\log_b \log_a t) = \Theta(\log \log t).$$