CSC 311 DATA STRUCTURES Fall'11

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1 Formal definition of Big-Oh and Big-Theta

Let $f : N \to R^+$ and $g : N \to R^+$, that is, f and g are functions (one may think of them as hypothetical running times of some programs) that take an integer n (the size of input) as an argument and return a positive real (a running time for an input of that size) as a value $f(n)$ or $g(n)$, respectively.

Definition 1.1

$$f \in O(g) \equiv \exists k \in R^+, \exists n_0 \in N, \forall n \geq n_0, f(n) \leq k \times g(n)$$

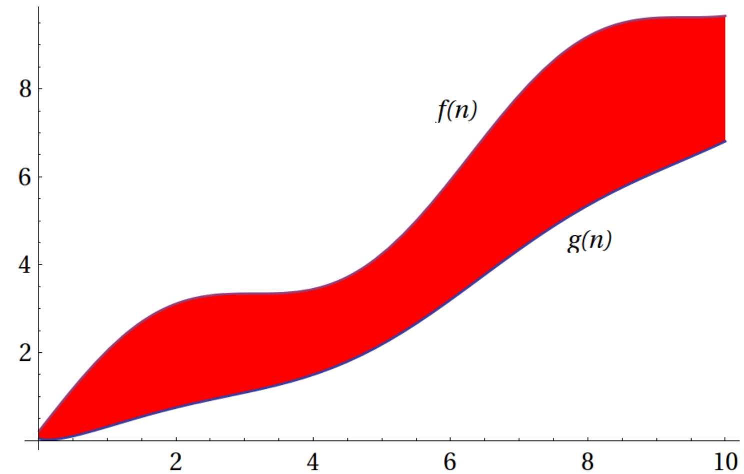

Figure 1: An example of f and g.

Definition 1.2

$$f \in \Theta(g) \equiv \exists k_1, k_2 \in R^+, \exists n_0 \in N, \forall n \geq n_0, k_1 \times g(n) \leq f(n) \leq k_2 \times g(n)$$

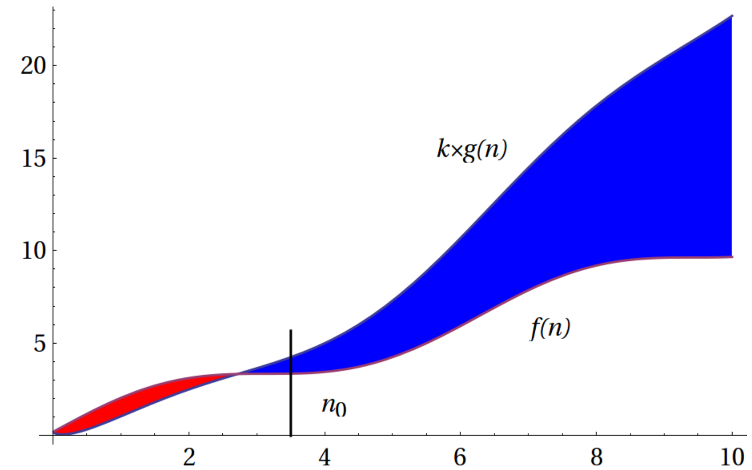

Figure 2: An example of k and n_0 that shows $f \in O(g)$.

Fact 1.3

$$f \in \Theta(g) \equiv f \in O(g) \wedge g \in O(f)$$

Fact 1.4 *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ *exists then*

$$f \in O(g) \equiv \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty.$$

Fact 1.5 *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ *exists then*

$$f \in \Theta(g) \equiv 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty.$$

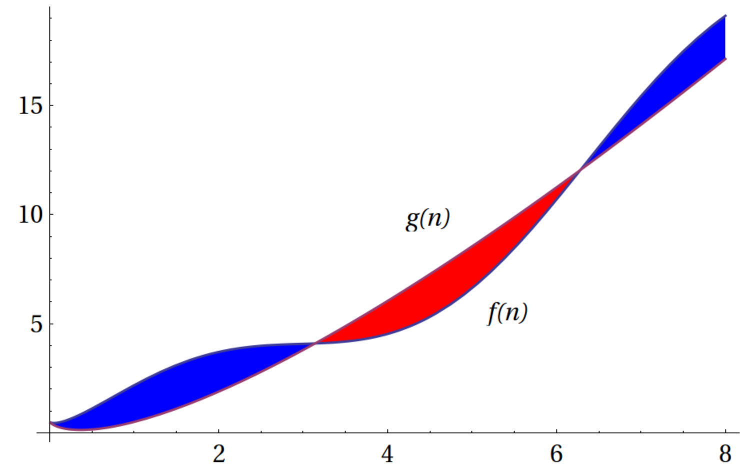

Figure 3: Another example of f and g.

In some cases, de l'Hôpital rule is a handy tool to compute limits of such fractions of differentiable functions. We quote it in a form that is useful for derivation of big Oh and big Theta facts.

Theorem 1.6 *Assume that f and g are differentiable functions,* $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$ *or* ∞*, and that the limit* $\lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ *exists. Then*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

Example 1.7 *We will show that* $n \log n \in O(n^2)$.

It suffices to show that $\lim_{n\to\infty} \frac{n \log n}{n^2} < \infty$. *Indeed,*

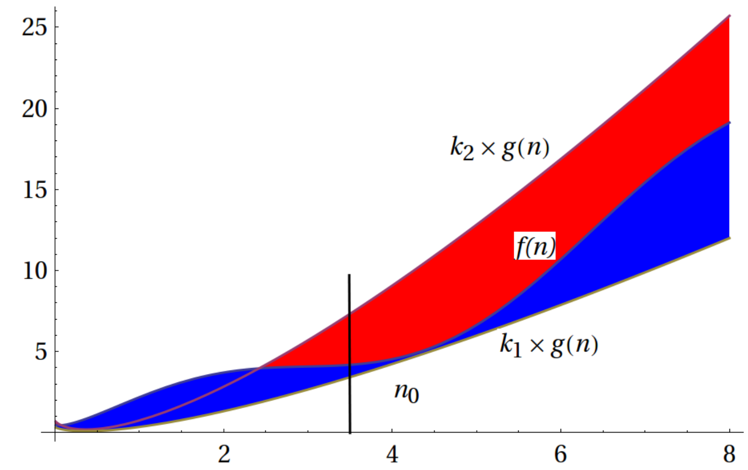

Figure 4: An example of k_1, k_2, and n_0 that show $f \in \Theta(g)$.

$$\lim_{n\to\infty} \frac{n \log n}{n^2} = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\log' n}{n'} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0 < \infty.$$

The following two facts are optional but very useful for students who use Mathematica.

Fact 1.8

$$f \in O(g) \equiv \overline{\lim_{n\to\infty}} \frac{f(n)}{g(n)} < \infty.$$

Fact 1.9

$$f \in \Theta(g) \equiv 0 < \varliminf_{n \to \infty} \frac{f(n)}{g(n)} \wedge \varlimsup_{n \to \infty} \frac{f(n)}{g(n)} < \infty.$$

2 Formal definition of little-oh (optional)

Definition 2.1

$$f \in o(g) \equiv \forall k \in R^+, \exists n_0 \in N, \forall n \geq n_0, f(n) \leq k \times g(n)$$

Fact 2.2

$$f \in o(g) \equiv \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Fact 2.3

$$f \in o(g) \Rightarrow f \in O(g) \wedge f \notin \Theta(g)$$

Note. The converse implication does not hold. Let

$$f(n) = \begin{cases} n \text{ if } n \text{ is odd} \\ \\ n^2 \text{ if } n \text{ is even.} \end{cases} \tag{1}$$

One can show (It's a very good exercise!) that $f \in O(n^2)$ and $n^2 \notin O(f)$. In particular, $n^2 \notin \Theta(f)$. However, $f \notin o(n^2)$.

Example 2.4 *Using the fact* $\lim_{n \to \infty} \frac{n \log n}{n^2} = 0$ *derived in Example 1.7, we conclude that* $n \log n \in o(n^2)$.