The # of N – permutations that result in the best BST for $N = 2^{H+1} - 1$ as a function S[H] of H

$$S[0] = 1 = \binom{0}{0} = \binom{2^1 - 2}{2^0 - 1}$$

$$S[1] = 2 = \binom{2}{1} = \binom{2^2 - 2}{2^1 - 1}$$

$$S[H] = \binom{2^{H+1} - 2}{2^H - 1} \times S[H-1]^2$$

The left and right subtrees are complete of depth H – 1. So each takes S[H – 1] number of $\frac{(N-1)}{2}$ – permutations to make it. Each combination results in a different N – permutation for the entire tree,
hence factor $S[H-1]^2$. $\binom{2^{H+1} - 2}{2^H - 1}$ is the number of ways how numbers < $\frac{(N+1)}{2}$ and numbers > $\frac{(N+1)}{2}$, each of these a choice of $2^H - 1$ out of total $2^{H+1} - 2$ $\left(\text{the root } \frac{(N+1)}{2} \text{ of the tree is in place already}\right)$ are mixed together.

$$S[H] = \binom{2^{H+1} - 2}{2^H - 1} \times S[H-1]^2 =$$

$$\binom{2^{H+1} - 2}{2^H - 1}^{2^0} \times \binom{2^H - 2}{2^{H-1} - 1}^{2^1} \times \binom{2^{H-1} - 2}{2^{H-2} - 1}^{2^2} \times \ldots \times \binom{2^{H+1-i} - 2}{2^{H-i} - 1}^{2^i} \times \ldots \times \binom{2^{H+1-(H-1)} - 2}{2^{H-(H-1)} - 1}^{2^{H-1}} =$$

$$\prod_{i=0}^{H-1} \text{Binomial}\left[2^{H+1-i} - 2, 2^{H-i} - 1\right]^{2^i} =$$

$$\left(2^{H+1} - 2\right)! \Big/ \left(\left(2^H - 1\right)!\right)^2 \times \left(\left(2^H - 2\right)!\right)^2 \Big/ \left(\left(2^{H-1} - 1\right)!\right)^4 \times$$

$$\left(\left(2^{H-1} - 2\right)!\right)^4 \Big/ \left(\left(2^{H-2} - 1\right)!\right)^8 \times \ldots \times \left(\left(2^{H+1-i} - 2\right)!\right)^{2^i} \Big/ \left(\left(2^{H-i} - 1\right)!\right)^{2^{i+1}} \times \ldots \times$$

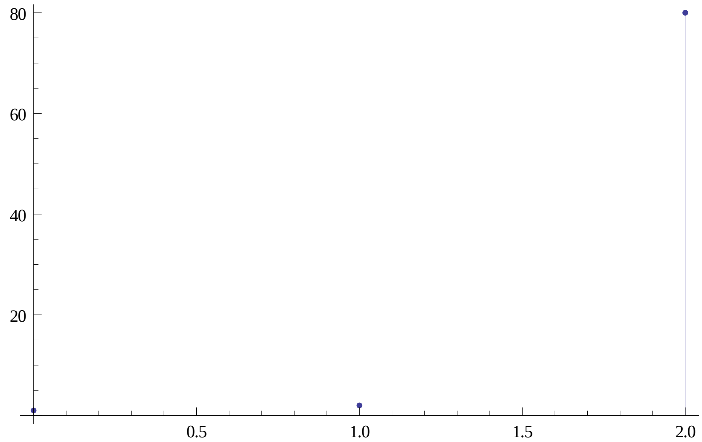
$$\left(\left(2^{H+1-(H-1)} - 2\right)!\right)^{2^{H-1}} \Big/ \left(\left(2^{H-(H-1)} - 1\right)!\right)^{2^H} = \prod_{i=0}^{H-1} \left(\left(2^{H+1-i} - 2\right)!\right)^{2^i} \Big/ \left(\left(2^{H-i} - 1\right)!\right)^{2^{i+1}} =$$

$$\left(2^{H+1} - 2\right)! \Big/ \left(\prod_{i=0}^{H-1} \left(2^{H-i} - 1\right)^{2^{i+1}}\right) =$$

$$\left(2^{H+1} - 2\right)! \Big/ \left(\prod_{i=0}^{H-2} \left(2^{H-i} - 1\right)^{2^{i+1}}\right) = \left(2^{H+1} - 2\right)! \Big/ \left(\prod_{i=1}^{H-1} \left(2^{H+1-i} - 1\right)^{2^i}\right)$$

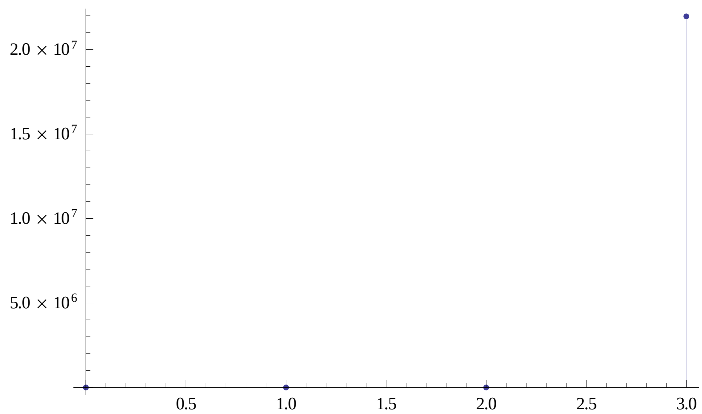
$$S[H] = \left(2^{H+1} - 2\right)! \Big/ \left(\prod_{i=1}^{H-1} \left(2^{H+1-i} - 1\right)^{2^i}\right)$$

```
DiscretePlot[(2^(H+1) - 2)! / (Product[(2^(H+1-i) - 1)^(2^i), {i, 1, H-1}]), {H, 0, 2}]
```



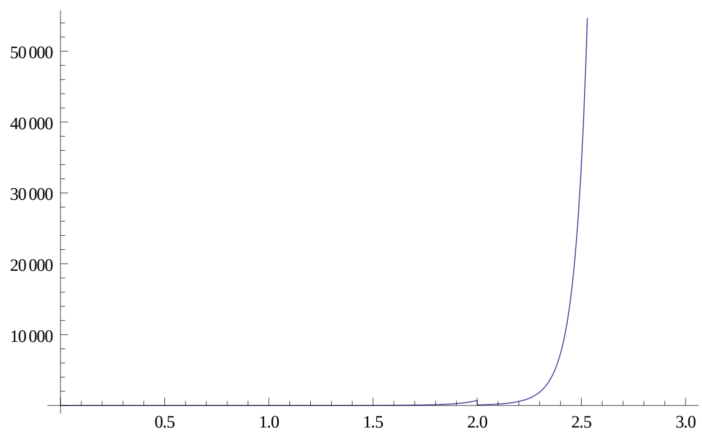
In[2]:=
```
DiscretePlot[(2^(H+1) - 2)! / (Product[(2^(H+1-i) - 1)^(2^i), {i, 1, H-1}]), {H, 0, 3}]
```

Out[2]=



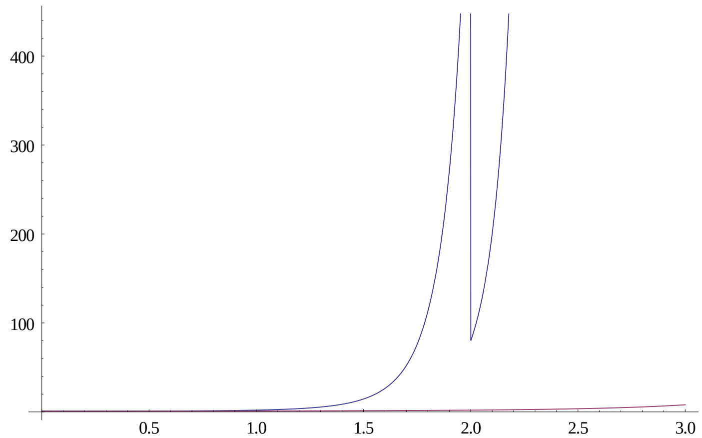
In[3]:=
```
Plot[(2^(H+1) - 2)! / (Product[(2^(H+1-i) - 1)^(2^i), {i, 1, H-1}]), {H, 0, 3}]
```

Out[3]=



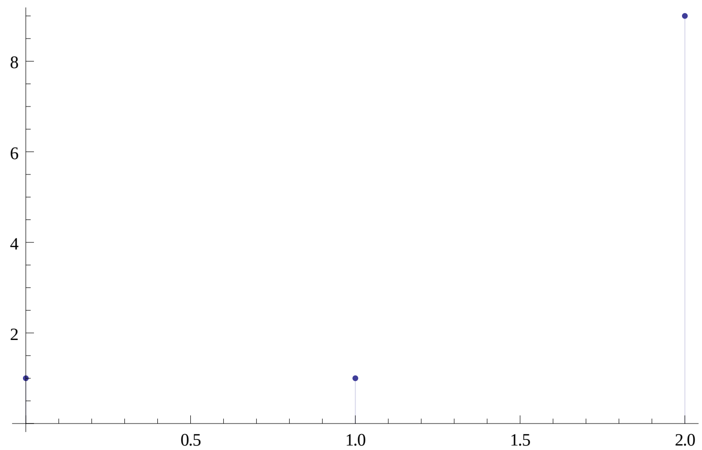
In[4]:= Plot$\left[\left\{\frac{(2^{H+1}-2)!}{\prod_{i=1}^{H-1}(2^{H+1-i}-1)^{2^i}},\ 2^{2^{H-1}-1}\right\},\ \{H, 0, 3\}\right]$

Out[4]=



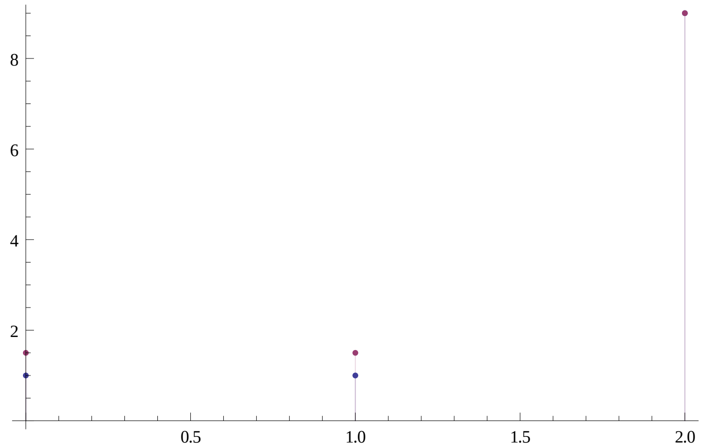

**Estimate of $\prod_{i=1}^{H-1}(2^{H+1-i}-1)^{2^i}$**

DiscretePlot$\left[\prod_{i=1}^{H-1}(2^{H+1-i}-1)^{2^i},\ \{H, 0, 2\}\right]$

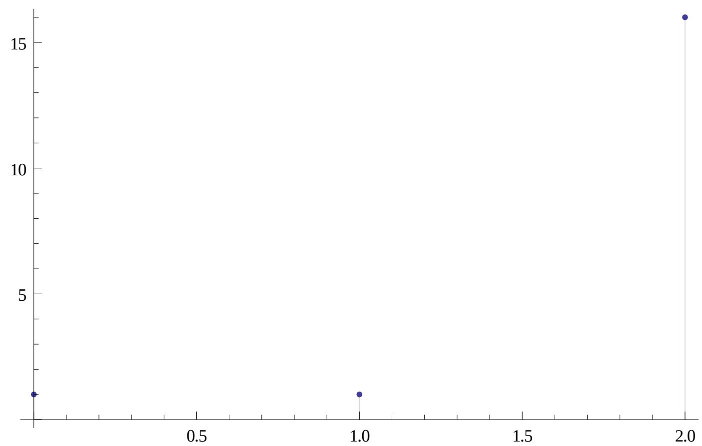



$$\prod_{i=1}^{H-1}(2^{H+1-i}-1)^{2^i} < 1+\frac{1}{2}\prod_{i=1}^{H-1}(2^{H+1-i})^{2^i}$$

**DiscretePlot**$\left[\left\{\prod_{i=1}^{H-1}\left(2^{H+1-i}-1\right)^{2^i},\ 1+\frac{1}{2}\prod_{i=1}^{H-1}\left(2^{H+1-i}\right)^{2^i}\right\},\ \{H,\ 0,\ 2\}\right]$



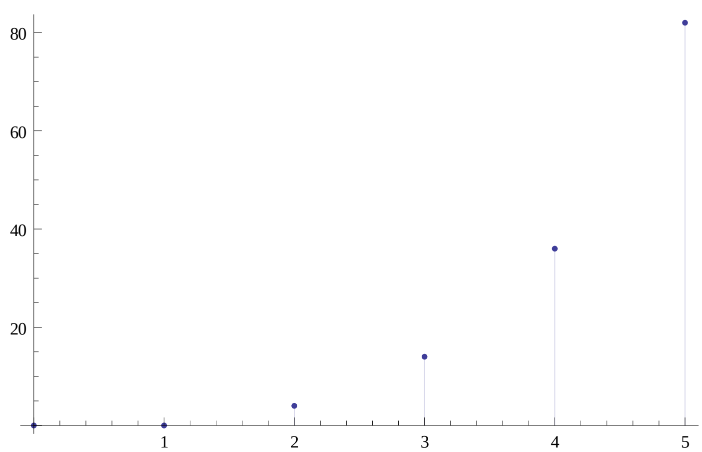
**DiscretePlot**$\left[\left\{\prod_{i=1}^{H-1}\left(2^{H+1-i}\right)^{2^i}\right\},\ \{H,\ 0,\ 2\}\right]$



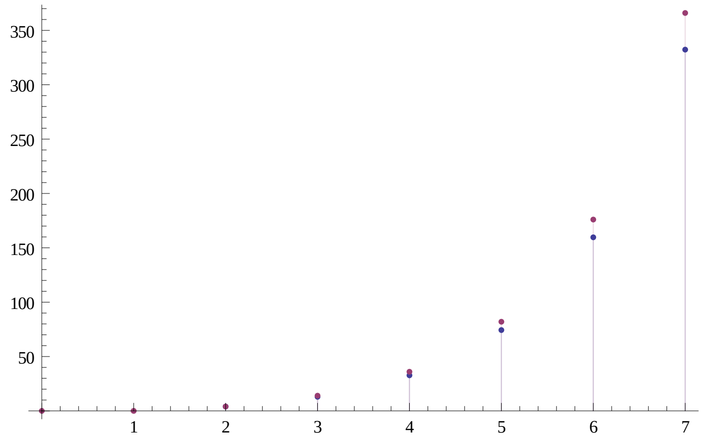
$F[H]=\prod_{i=1}^{H-1}\left(2^{H+1-i}\right)^{2^i}$

**Log2**$[F[H]]=$ **Log2**$\left[\prod_{i=1}^{H-1}\left(2^{H+1-i}\right)^{2^i}\right]$

**DiscretePlot**$\left[\left\{\text{Log2}\left[\prod_{i=1}^{H-1}\left(2^{H+1-i}\right)^{2^i}\right]\right\},\ \{H,\ 0,\ 5\}\right]$

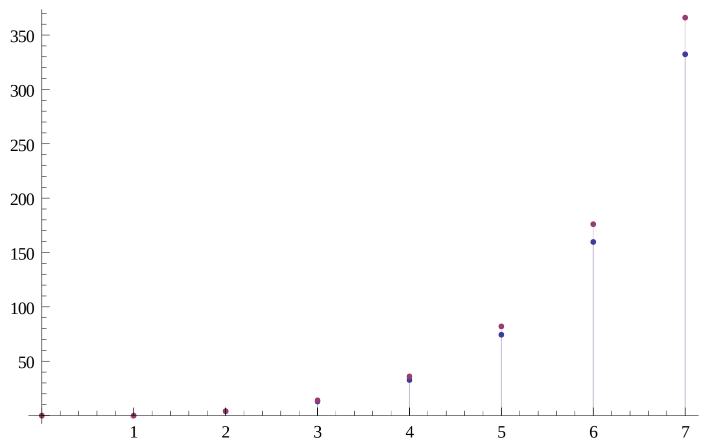


$$\text{DiscretePlot}\left[\left\{\text{Log2}\left[\prod_{i=1}^{H-1}\left(2^{H+1-i}-1\right)^{2^i}-1\right]+1,\ \text{Log2}\left[\prod_{i=1}^{H-1}\left(2^{H+1-i}\right)^{2^i}\right]\right\},\ \{H,0,7\}\right]$$



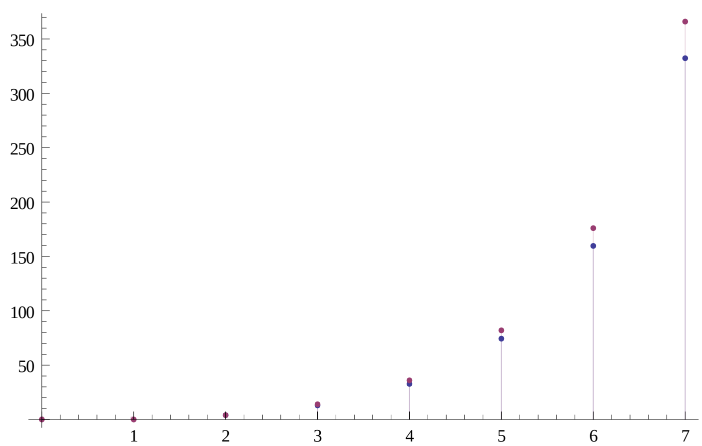
$$\sum_{i=1}^{H-1}\text{Log2}\left[\left(2^{H+1-i}\right)^{2^i}\right]$$

$$\text{DiscretePlot}\left[\left\{\text{Log2}\left[\prod_{i=1}^{H-1}\left(2^{H+1-i}-1\right)^{2^i}-1\right]+1,\ \sum_{i=1}^{H-1}\text{Log2}\left[\left(2^{H+1-i}\right)^{2^i}\right]\right\},\ \{H,0,7\}\right]$$



$$\sum_{i=1}^{H-1}2^i\ \text{Log2}\left[2^{H+1-i}\right]$$

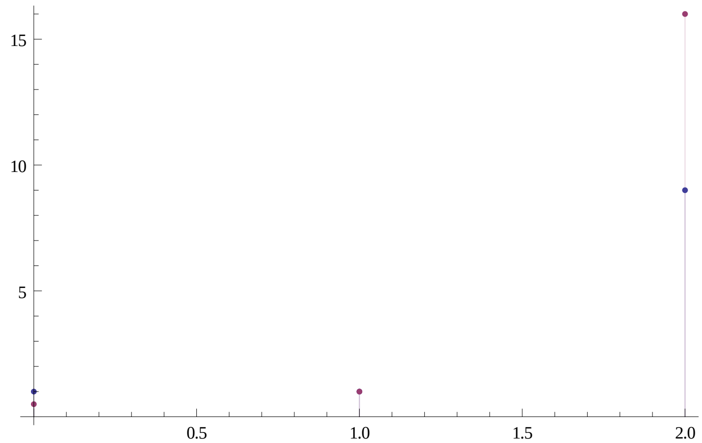
$$\text{DiscretePlot}\left[\left\{\text{Log2}\left[\prod_{i=1}^{H-1}\left(2^{H+1-i}-1\right)^{2^i}-1\right]+1,\ \sum_{i=1}^{H-1}2^i\ (H+1-i)\right\},\ \{H,0,7\}\right]$$



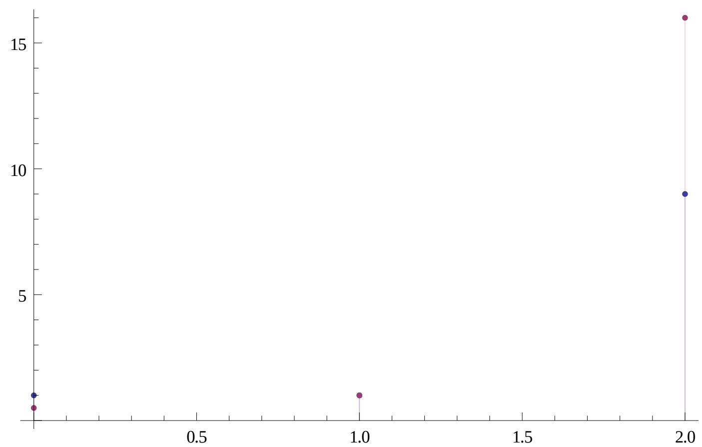
$\sum_{i=1}^{H-1} 2^i \ (H+1-i)$

$-4 + 3 \times 2^H - 2\ H$

DiscretePlot$\left[\left\{\prod_{i=1}^{H-1}\left(2^{H+1-i}-1\right)^{2^i},\ 2^{-4+3\times 2^H-2\ H}\right\},\ \{H,\ 0,\ 2\}\right]$

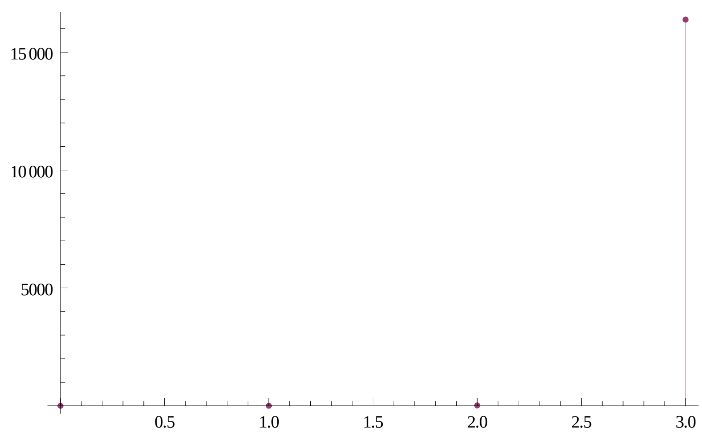



DiscretePlot$\left[\left\{\prod_{i=1}^{H-1}\left(2^{H+1-i}-1\right)^{2^i},\ 2^{3\times 2^H-2\ H-4}\right\},\ \{H,\ 0,\ 2\}\right]$



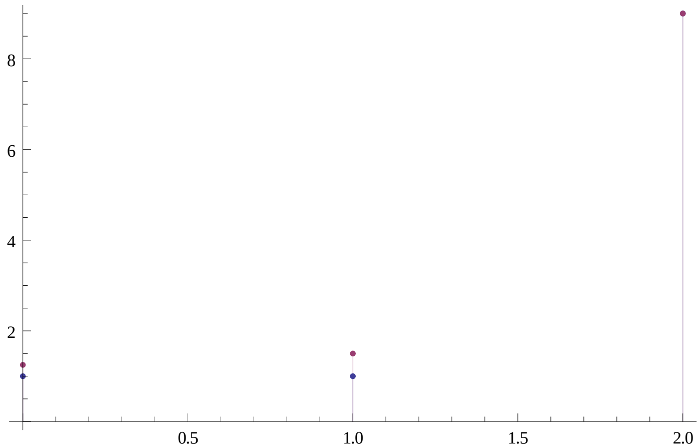

$F[H] = 2^{3\times 2^H-2\ H-4}$

DiscretePlot$\left[\left\{\prod_{i=1}^{H-1}\left(2^{H+1-i}\right)^{2^i},\ 2^{3\times 2^H-2\ H-4}\right\},\ \{H,\ 0,\ 3\}\right]$



$$\prod_{i=1}^{H-1} \left(2^{H+1-i} - 1\right)^{2^i} < 1 + \frac{1}{2}\, 2^{3\times 2^H - 2H - 4}$$

$$\text{DiscretePlot}\left[\left\{\prod_{i=1}^{H-1} \left(2^{H+1-i} - 1\right)^{2^i},\ 1 + \frac{1}{2}\, 2^{3\times 2^H - 2H - 4}\right\},\ \{H, 0, 2\}\right]$$



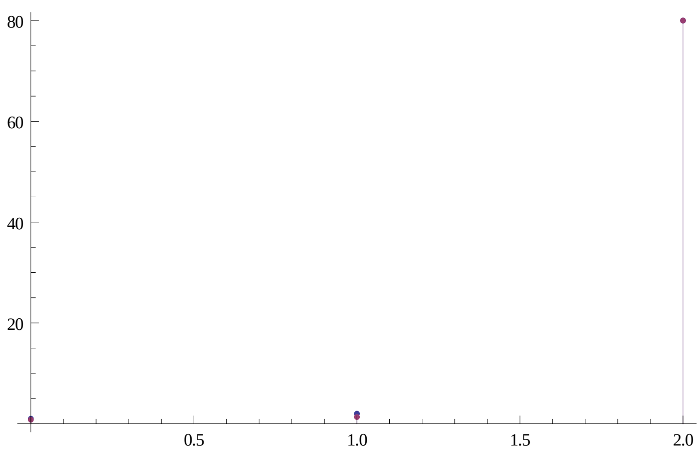

$$S[H] > \left(2^{H+1} - 2\right)! \Big/ \left(\prod_{i=1}^{H-1} \left(2^{H+1-i} - 1\right)^{2^i}\right) > \left(2^{H+1} - 2\right)! \Big/ \left(1 + \frac{1}{2}\, 2^{3\times 2^H - 2H - 4}\right) =$$

$$(N - 1)! \Big/ \left(1 + \frac{1}{2}\, 2^{3\,\frac{N+1}{2} - 2\,(\text{Log2}[N+1] - 1) - 4}\right) =$$

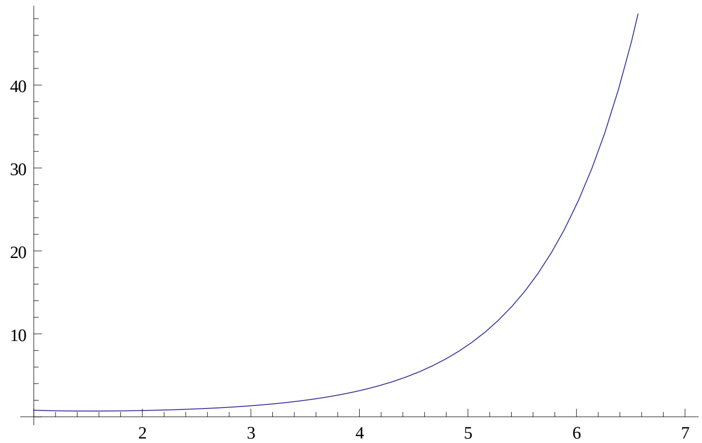
$$(N - 1)! \Big/ \left(1 + 2^{3\,\frac{N+1}{2} - 2\,(\text{Log2}[N+1] - 1) - 5}\right) = (N - 1)! \Big/ \left(1 + 2^{3\,\frac{N+1}{2} - 2\,(\text{Log2}[N+1] - 1) - 5}\right) =$$

$$\textbf{ERROR} = \left((N - 1)!\ 2^{2\,\text{Log2}[N+1]}\right) \Big/ \left(1 + 2^{3\,\frac{N+1}{2} - 3}\right) = \left((N - 1)!\ (N + 1)^2\right) \Big/ \left(1 + 2^{3\,\frac{N+1}{2} - 3}\right)$$

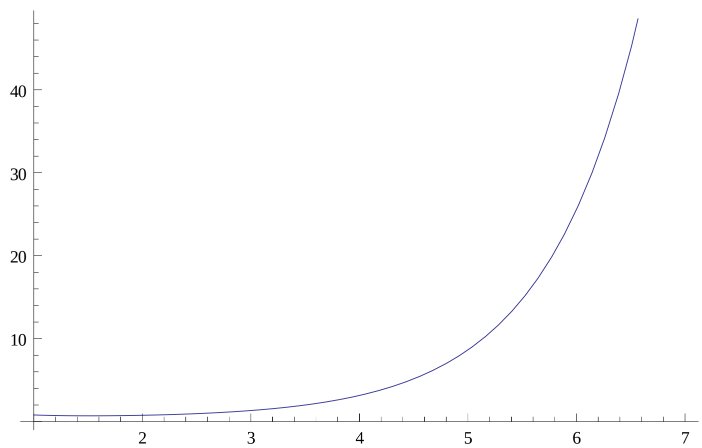
$$\text{DiscretePlot}\left[\left\{\left(2^{H+1} - 2\right)! \Big/ \left(\prod_{i=1}^{H-1} \left(2^{H+1-i} - 1\right)^{2^i}\right),\ \left(2^{H+1} - 2\right)! \Big/ \left(1 + \frac{1}{2}\, 2^{3\times 2^H - 2H - 4}\right)\right\},\ \{H, 0, 2\}\right]$$



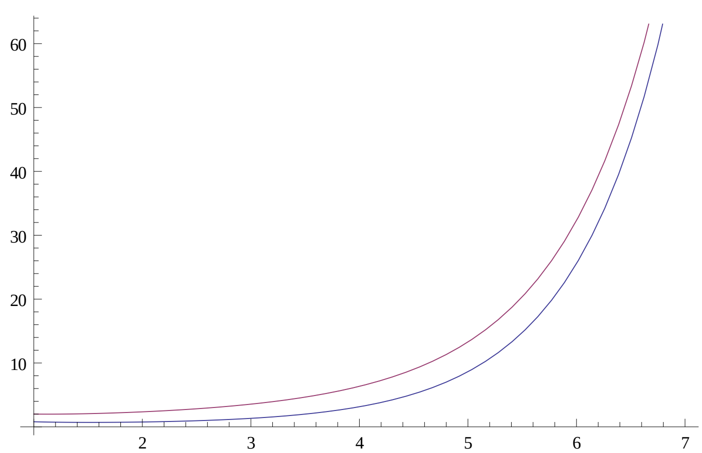
$$\text{Plot}\left[\left\{(N-1)!\Big/\left(1+\frac{1}{2}\,2^{3\,\frac{N+1}{2}-2\,(\text{Log2}[N+1]-1)-4}\right)\right\},\ \{N,1,7\}\right]$$




$$\text{Plot}\left[\left\{(N-1)!\Big/\left(1+2^{3\,\frac{N+1}{2}-2\,(\text{Log2}[N+1]-1)-5}\right)\right\},\ \{N,1,7\}\right]$$




$$\text{Plot}\left[\left\{(N-1)!\Big/\left(1+2^{3\,\frac{N+1}{2}-2\,(\text{Log2}[N+1]-1)-5}\right),\ \left((N-1)!\;2^{2\,\text{Log2}[N+1]}\right)\Big/\left(1+2^{3\,\frac{N+1}{2}-3}\right)\right\},\ \{N,1,7\}\right]$$



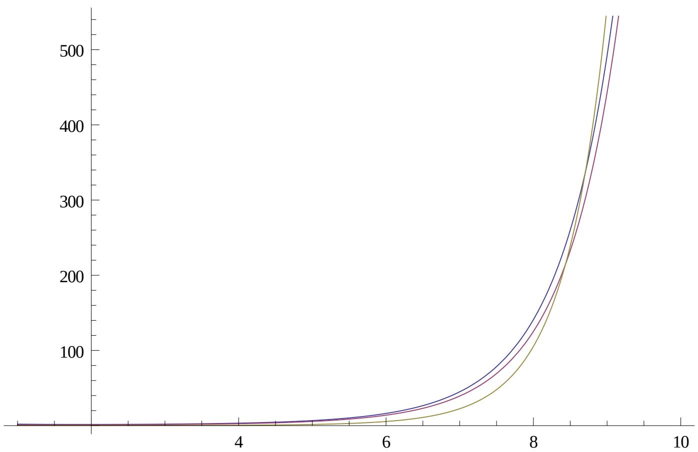
$$S[H] > \left(2^{H+1}-2\right)! \Big/ \left(\prod_{i=1}^{H-1}\left(2^{H+1-i}-1\right)^{2^i}\right) > \frac{\left(2^{H+1}-2\right)!}{2^{3\times 2^H-2H-4}} = (N-1)! \Big/ 2^{3\,\frac{N+1}{2}-2\,(\text{Log2}[N+1]-1)-4} =$$

$$(N-1)! \Big/ 2^{3\,\frac{N+1}{2}-2\,(\text{Log2}[N+1]-1)-4} = \left((N-1)!\;2^{2\,\text{Log2}[N+1]}\right) \Big/ 2^{3\,\frac{N+1}{2}-2} =$$

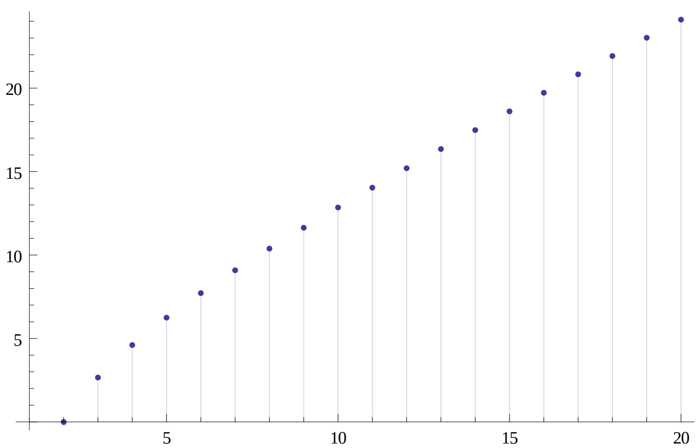
$$\left((N-1)!\;(N+1)^2\right) \Big/ 2^{3\,\frac{N+1}{2}-2} = \left(4\,(N-1)!\;(N+1)^2\right) \Big/ 2^{3\,\frac{N+1}{2}} =$$

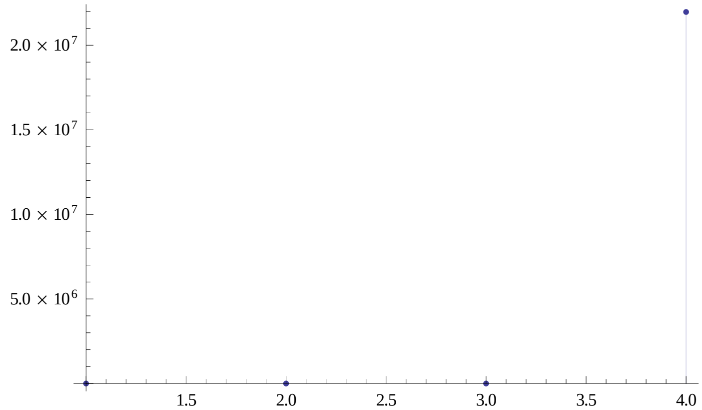
$$\left(4\,(N-1)!\;(N+1)^2\right) \Big/ \left(2^{\frac{3}{2}}\right)^{N+1} > \left(4\,(N-1)!\;(N+1)\,N\right) \Big/ \left(2^{\frac{3}{2}}\right)^{N+1} = \left(4\,(N+1)!\right) \Big/ \left(2^{\frac{3}{2}}\right)^{N+1}$$

$$\text{Plot}\left[\left\{(N-1)! \Big/ 2^{3\,\frac{N+1}{2}-2\,(\text{Log2}[N+1]-1)-4},\ \frac{4\,(N+1)!}{\left(2^{\frac{3}{2}}\right)^{N+1}},\ \frac{(N+1)!}{N\times 2^{\frac{3N+11}{4}}}\right\},\ \{N,1,10\}\right]$$

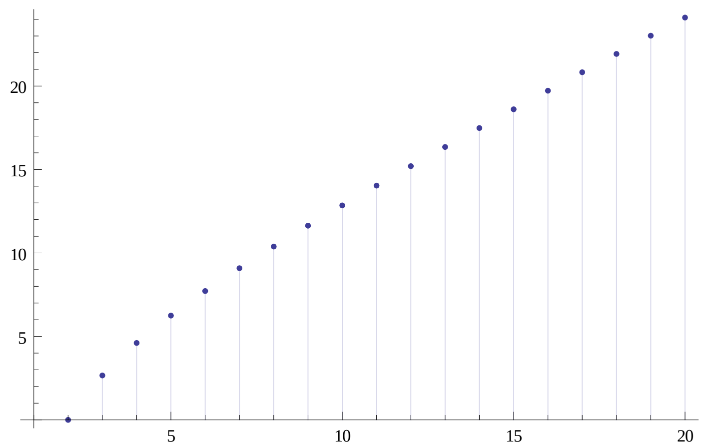



$$\text{DiscretePlot}\left[\text{Log2}\left[\text{Log2}\left[\prod_{i=0}^{n-1}\text{Binomial}\left[2^{n-i}-2,\ 2^{n-i-1}-1\right]^{2^i}\right]\right],\ \{n,1,20\}\right]$$

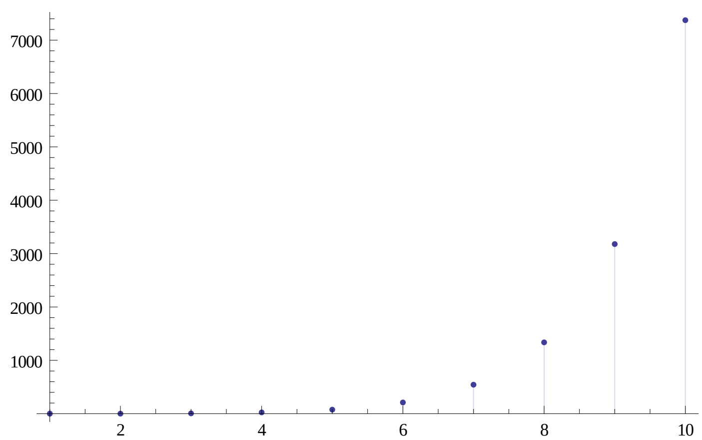





**DiscretePlot** $\left[\text{Log2}\left[\text{Log2}\left[(2^{n}-2)!\Big/\left(\prod_{i=1}^{n-1}\left(2^{n-i}-1\right)^{2^{i}}\right)\right]\right],\ \{n,\ 1,\ 20\}\right]$




**DiscretePlot** $\left[\text{Log2}\left[(2^{n}-2)!\Big/\left(\prod_{i=1}^{n-1}\left(2^{n-i}-1\right)^{2^{i}}\right)\right],\ \{n,\ 1,\ 10\}\right]$




$\sum_{i=1}^{n-1}(n-i)\,2^{i}$

$2\,(-1+2^{n}-n)$