Logic Programming vs Mathematical Logic

Marek A. Suchenek
California State University – Dominguez Hills
Carson, CA 90747

Goals

• To define logic programs.

• To define various concepts of models for propositional logic programs.

• To define various interpretations of negation in propositional logic programs.

All the above for a propositional language $L$ with $N$ propositional variables.
1 Clauses and programs

A literal is a propositional variable or a negated propositional variable.

E.g., $A$ and $\neg B$ are literals while $A \lor \neg B$ is not.

A clause carrier – a set of literals with every propositional variable occurring at most once.

E.g.,

$$\varphi = \{A, \neg B\}$$
$$\psi = \{B, C\}$$
$$\rho = \{B, \neg C, \neg D\}$$
$$\gamma = \{\neg B\}$$

A program carrier – a set of clause carriers.

E.g.,

$$\Phi = \{\varphi, \psi\} =$$
$$\{\{A, \neg B\}, \{B, C\}\}$$

$$\Psi = \{\varphi, \rho\} =$$
$$\{\{A, \neg B\}, \{B, \neg C, \neg D\}\}$$
A *clause* – a formula of the form $\lor p$, where $p$ is a clause carrier.

E.g.,

$$\lor \varphi = A \lor \neg B =$$

$$A \leftarrow B$$

$$\lor \psi = B \lor C$$

$$\lor \rho = B \lor \neg C \lor \neg D =$$

$$B \leftarrow (C \land D)$$

$$\lor \gamma = \neg B =$$

$$\leftarrow B$$

The part on the left-hand side of $\leftarrow$ in a clause is called a *head*, while the part on the right-hand side of $\leftarrow$ in a clause is called a *body*.

So, the clause $\lor \psi$ has no body and is sometimes called a *fact* or an *assertion*. 
The clause \( \lor \gamma \) has no head and is called a *query*.

**Note.** Usually, the \( \land \)'s in clauses in \( \leftarrow \) form are replaced with colons and the parentheses are omitted for notational convenience.

For instance,

\[
A \leftarrow (C \land D)
\]

is usually written as:

\[
A \leftarrow C, D
\]

Moreover, \( \leftarrow \) is often typed as \( - \), like, for instance, in

\[
A : - C, D
\]
A \textit{program} – a formula of the form $\land \lor \Phi$.

E.g.,

\[
\land \lor \Phi = (\lor \varphi) \land (\lor \psi) = (A \leftarrow B) \land (B \lor C)
\]

\[
\land \lor \Psi = (\lor \varphi) \land (\lor \rho) = (A \lor \neg B) \land (B \lor \neg C \lor \neg D)
\]

\[
(A \leftarrow B) \land (B \leftarrow C, D)
\]
A Horn clause – a formula of the form $\forall p$, where $p$ is a clause carrier with at most one positive literal.

E.g.,

$$\forall \varphi = A \lor \neg B = \quad A \leftarrow B$$

$$\forall \rho = B \lor \neg C \lor \neg D = \quad B \leftarrow C, D$$

and

$$\forall \gamma = \neg B = \quad \leftarrow B$$

are Horn clauses while

$$\forall \psi = B \lor C$$

is not.
A Horn program – a formula of the form $\land \lor \Phi$ where $\Phi$ is a set of Horn clause carriers.

E.g.,

$$\land \lor \Psi = (\lor \varphi) \land (\lor \rho) =$$

$$(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$$

$$(A \leftarrow B) \land (B \leftarrow C, D)$$

is a Horn program while

$$\land \lor \Phi = (\lor \varphi) \land (\lor \psi) =$$

$$(A \leftarrow B) \land (B \lor C')$$

is not.
2 Propositional models

Propositional model – a clause carrier that contains all propositional variables.

For instance, if $L$ contains only four propositional variables, $A, B, C, D$, then the set

$$M = \{\neg A, \neg B, C, \neg D\}$$

is a model. Its intention is to assign truth values to the variables $A, B, C, D$. In particular, it assigns value true to variable $C$ and value false to variables $A, B, D$.

If $p$ is a clause or a program then $M$ is a propositional model of $p$ iff the truth assignment that $M$ defines makes all the clauses in $p$ true.
E.g., $M$ is a propositional model of

$$\land \lor \Psi = (\lor \varphi) \land (\lor \rho) =$$

$$(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$$

$$(A \leftarrow B) \land (B \leftarrow C, D)$$

The set of all propositional models of $p$ is denoted by $\text{Mod}(p)$.

The \textit{closed-world assumption} reconstructs derives negative literals from their absence.

E.g., $\text{cwa}\{C\} = \{\neg A, \neg B, C, \neg D\}$.

Application of $\text{cwa}$ allows to skip all negative literals in a propositional model. We will be doing just that.
Example.

\[ Mod((A \leftarrow B) \land (B \leftarrow C,D)) = \]

= \{\}, \{A\}, \{C\}, \{D\}, \{A, B\}, \{A, C\}, \{A, D\},

\{A, B, C\}, \{A, B, D\}, \{A, B, C, D\}

A propositional model \(M\) of \(p\) is \textit{minimal} iff no other model of \(p\) is a proper subset of \(M\).

For instance, \(\{\}\) is the only minimal model of \((A \leftarrow B) \land (B \leftarrow C, D)\).
Example. Let's find all minimal models of this logic program $P$:

\[
\begin{align*}
A & \leftarrow B \\
B & \leftarrow C, D \\
B & \\
D
\end{align*}
\]

Here there are all models of $P$:

\{
A, B, D\}, \{A, B, C, D\}

Out of those, only \{A, B, D\} is minimal.
Notation. If $P$ is a logic program the $\text{Mod}_{\text{min}}(P)$ denotes the set of all minimal models of $P$.

Definition. $\text{Mod}_{\text{min}}(P)$ is the semantics of a logic program $P$. A query (a headless clause) $\gamma$ to $P$ yields an answer “yes” iff $\gamma$ is true in every minimal model of $P$.

Fact. If $P$ is a Horn program then $\text{Mod}_{\text{min}}(P)$ has only one element.

Example

$$\varphi = \{A_1 \lor \ldots \lor A_n, B_1 \lor \ldots \lor B_n\},$$

where $A_i$’s and $B_j$’s are distinct variables, has only $n^2$ minimal models, but it has $(2^n - 1)^2$ models.
The pigeonhole principle (PHP).

\[ \text{PHP}^- \]
\[
\{\{A_{1,1}, \ldots, A_{1,n}\}, \ldots, \{A_{n+1,1}, \ldots, A_{n+1,n}\}\}
\]

\[ \text{PHP}^+ \]
\[
\{\{A_{1,1}, A_{2,1}\}, \{A_{1,1}, A_{3,1}\}, \ldots, \{A_{n,1}, A_{n+1,1}\}, \ldots, \{A_{1,n}, A_{2,n}\}, \{A_{1,n}, A_{3,n}\}, \ldots, \{A_{n,n}, A_{n+1,n}\}\}
\]

\[ \land \lor \text{PHP}^- \vdash \lor \land \text{PHP}^+ \]
\[ \land \lor \text{PHP}^- \vdash_{\text{min}} \lor \land \text{PHP}^+ \]

\[ \land \lor \text{PHP}^- \] has \( N = n \times (n + 1) \) variables and roughly \( 2^N \) models.

The number of minimal models of \( \land \lor \text{PHP}^- \) is \( n^{n+1} \).

The ratio is:
\[
\frac{2^N}{n^{n+1}} \approx \left(\frac{2}{\sqrt[n]{n}}\right)^N \geq (1.99)^N
\]
The pigeonhole principle reformulated

\(\land \lor PHP^-\)

\((A_{1,1} \lor ... \lor A_{1,n})\land \ldots \land (A_{n+1,1} \lor ... \lor A_{n+1,n})\)

\(\lor \land PHP^+\)

\((A_{1,1} \land A_{2,1}) \lor (A_{1,1} \land A_{3,1}) \lor ... \lor (A_{n,1} \land A_{n+1,1})\lor \ldots \lor (A_{1,n} \land A_{2,n}) \lor (A_{1,n} \land A_{3,n}) \lor ... \lor (A_{n,n} \land A_{n+1,n})\)

\(\land \lor PHP^- \vdash \lor \land PHP^+\)

\(\land \lor PHP^- \vdash_{\text{min}} \lor \land PHP^+\)
The special case: 3 pigeons, 2 pigeonholes

\((PHP_2)\).

\[\land \lor PHP^-_2\]
\[= (A_{1,1} \lor A_{1,2}) \land \]
\[\land (A_{2,1} \lor A_{2,2}) \land \]
\[\land (A_{3,1} \lor A_{3,2})\]

\[\lor \land PHP^+_2\]
\[= (A_{1,1} \land A_{2,1}) \lor (A_{1,1} \land A_{3,1}) \lor (A_{2,1} \land A_{3,1}) \lor (A_{1,2} \land A_{2,2}) \lor (A_{1,2} \land A_{3,2}) \lor (A_{2,2} \land A_{3,2})\]

\[\land \lor PHP^-_2 \vdash \lor \land PHP^+_2\]

\[\land \lor PHP^-_2 \vdash_{\text{min}} \lor \land PHP^+_2\]

\(\land \lor PHP^-_2\) has \(N = 2 \times 3 = 6\) variables and \(\frac{3}{4} \times 2^6 = 48\) models.

The number of \textit{minimal} models of \(\land \lor PHP^-_2\) is \(n^{n+1} = 2^3 = 8\).