CSC 501/401 ANALYSIS ALG Sp '13

March 17, 2014

1 Formal definition of Big-Oh and Big-Theta

Let $f : N \to R^+$ and $g : N \to R^+$, that is, f and g are functions (one may think of them as hypothetical running times of some programs) that take an integer n (the size of input) as an argument and return a positive real (a running time for an input of that size) as values $f(n)$ or $g(n)$, respectively.

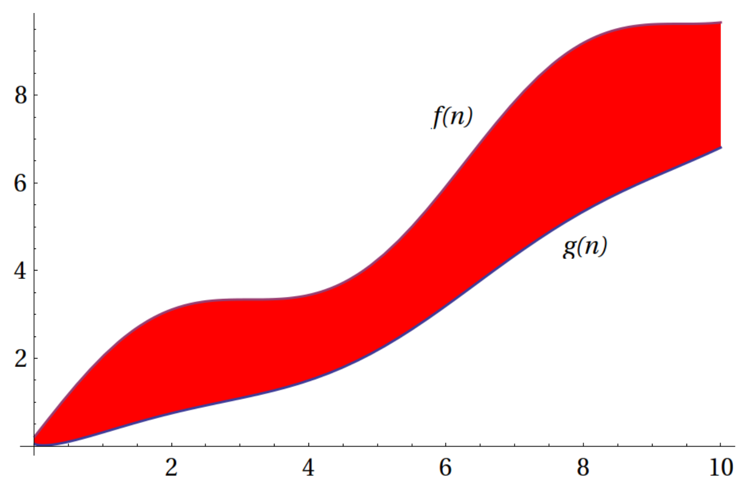

Figure 1: An example of f and g.

Definition 1.1

$$f \in O(g) \equiv \exists k \in R^+, \exists n_0 \in N, \forall n \geq n_0, f(n) \leq k \times g(n)$$

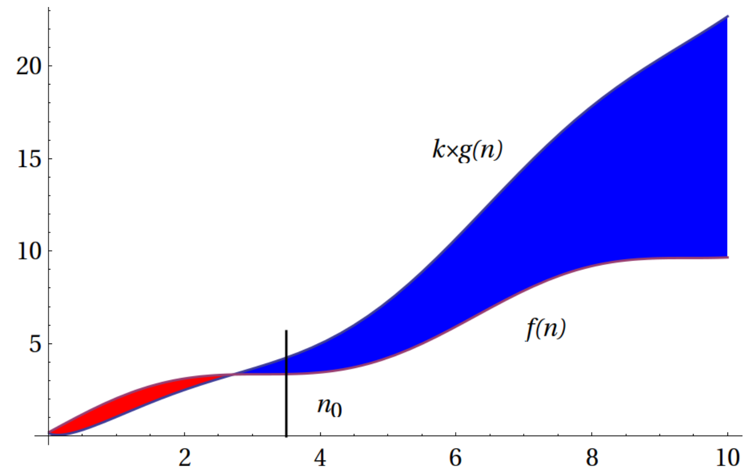

Figure 2: An example of k and n_0 that shows $f \in O(g)$.

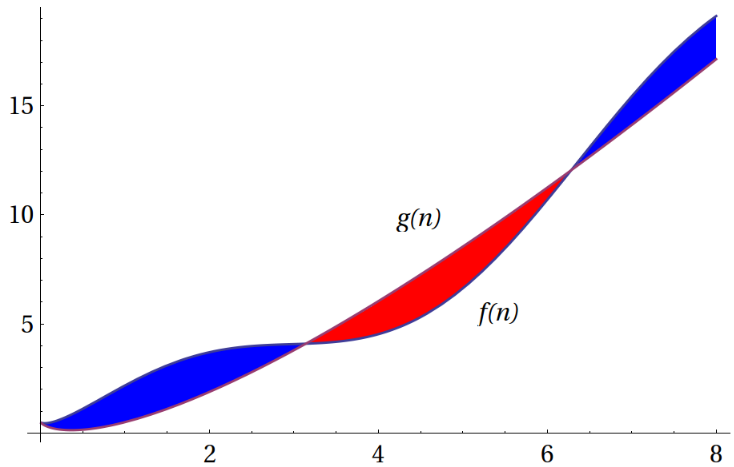

Figure 3: Another example of f and g.

Definition 1.2

$$f \in \Theta(g) \equiv \exists k_1, k_2 \in R^+, \exists n_0 \in N, \forall n \geq n_0, k_1 \times g(n) \leq f(n) \leq k_2 \times g(n)$$

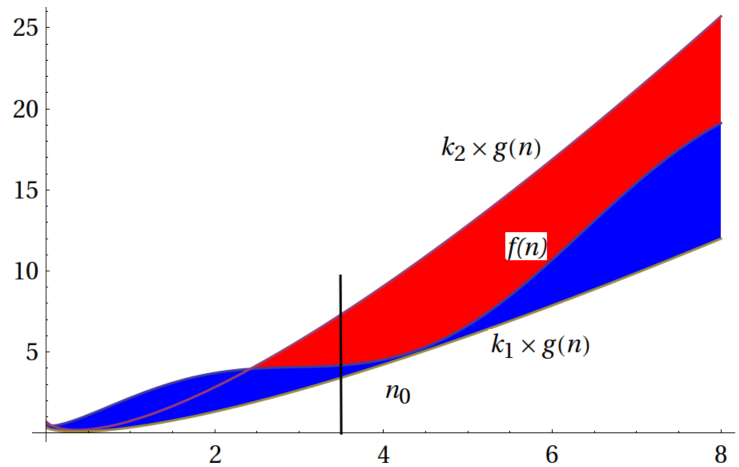

Figure 4: An example of k_1, k_2, and n_0 that show $f \in \Theta(g)$.

Fact 1.3

$$f \in \Theta(g) \equiv f \in O(g) \wedge g \in O(f)$$

Fact 1.4 *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ *exists then*

$$f \in O(g) \equiv \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty.$$

Fact 1.5 *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ *exists then*

$$f \in \Theta(g) \equiv 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty.$$

In some cases, de l'Hôpital rule is a handy tool to compute limits of such fractions of differentiable functions. We quote it in a form that is useful for derivation of big Oh and big Theta facts.

Theorem 1.6 *Assume that f and g are differentiable functions,* $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$ *or* ∞*, and that the limit* $\lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ *exists. Then*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

Example 1.7 *We will show that* $n \log n \in O(n^2)$.

It suffices to show that $\lim_{n\to\infty} \frac{n \log n}{n^2} < \infty$. *Indeed,*

$$\lim_{n\to\infty} \frac{n \log n}{n^2} = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\log' n}{n'} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0 < \infty.$$

The following two facts are mandatory for graduate students and optional for undergraduete students.

Fact 1.8

$$f \in O(g) \equiv \overline{\lim_{n\to\infty}} \frac{f(n)}{g(n)} < \infty.$$

Fact 1.9

$$f \in \Theta(g) \equiv 0 < \underline{\lim_{n\to\infty}} \frac{f(n)}{g(n)} \wedge \overline{\lim_{n\to\infty}} \frac{f(n)}{g(n)} < \infty.$$

2 Formal definition of little-oh

Definition 2.1

$$f \in o(g) \equiv \forall K \in R^+, \exists n_0 \in N, \forall n \geq n_0, f(n) \leq K \times g(n)$$

Fact 2.2

$$f \in o(g) \equiv \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

3 Formal definition of Big-Omega

Definition 3.1

$$f \in \Omega(g) \equiv g \in O(f)$$

Fact 3.2

$$f \in \Omega(g) \equiv \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$