

CSC 401 ANALYSIS OF ALG'S Sp '14

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1 Some rates of growth of running time $T(n)$ and corresponding rates of growth of max input's size $Max(t)$ and its derivative $Max'(t)$

Let $T(n)$ be a running time of some program P. Let us assert that $T(n)$ is a growing function. This assertion holds in what may be considered a typical situation: the larger the input to program P the longer it takes to process it.

Under the above assertion, $Max(t)$, defined as the maximum size n of input for which $T(n) \leq t$, is the inverse of $T(n)$, that is,

$$t = T(n) \text{ iff } n = Max(t). \quad (1)$$

In particular, $Max(t)$ is a growing function as well. (Proof left as an easy exercise for the reader.)

The following fact holds for every differentiable growing function f :

$$(f_{inverse})'(x) = \frac{1}{f'(f_{inverse}(x))}, \quad (2)$$

where $f_{inverse}$ is the inverse of f (it exists since f is a growing function) and f' is the derivative of f . (Proof left as an intermediate exercise for the reader - requires calculus. *Hint*: Use the geometric/trigonometric interpretation of derivative.)

In particular,

$$Max'(t) = \frac{1}{T'(Max(t))}. \quad (3)$$

The derivative $Max'(t)$ of $Max(t)$ seems like a good measure of return on investment of a faster computer (or - equivalently - longer wait) for program P . It tells how fast (or slow) the maximum size of tractable input to P will grow with the increase of the computer's speed. So, the larger $Max'(t)$ the more cost effective it is at point t to run P on a faster computer. And vice versa: the smaller $Max'(t)$ the more wasteful it is at point t to run P on a faster computer.

When the measure $Max'(t)$ is decreasing then it might be insightful to consider also the reciprocal $\frac{1}{Max'(t)}$ of $Max'(t)$ as a measure of the cost of enlargement of maximum size $Max(t)$ of tractable input to program P . In such a case, it seems easier to evaluate visually the rate of growth of $\frac{1}{Max'(t)}$ than the rate of decline of $Max'(t)$ based on their graphs. The measure $\frac{1}{Max'(t)}$ tells how much faster (or longer) the program P must be executed in order to accomplish the unit increase of tractable input to P . So, the larger the $\frac{1}{Max'(t)}$ the more costly it is at point t to run P on even a slightly larger input.

By (3),

$$\frac{1}{Max'(t)} = T'(Max(t)). \quad (4)$$

Finding the Θ characterization of $f'(t)$ and of $\frac{1}{f'(t)}$ from the Θ characterization of $f(t)$ requires some extra assumption that f and its Θ benchmark (representative) satisfy assumptions of the de l'Hôpital rule.

Theorem 1.1 *Let f and g be positive, increasing, differentiable functions that both converge to 0 or both diverge to ∞ as their arguments diverge to ∞ . Assume that $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$ exists. Then*

$$f \in \Theta(g) \equiv f' \in \Theta(g') \equiv \frac{1}{f'} \in \Theta\left(\frac{1}{g'}\right).$$

Proof (optional for all students). $f' \in \Theta(g')$ iff [because $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$ exists] $0 < \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} < \infty$ iff (by de l'Hôpital rule) $0 < \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty$ iff [because $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ exists] iff $f \in \Theta(g)$. This completes the proof of the first equivalence.

$f' \in \Theta(g')$ iff [because $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$ exists] $0 < \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} < \infty$ iff $0 < \lim_{x \rightarrow \infty} \frac{\frac{g'(x)}{f'(x)}}{\frac{1}{g'(x)}} < \infty$ iff $0 < \lim_{x \rightarrow \infty} \frac{\frac{f'(x)}{1}}{\frac{f'(x)}{g'(x)}} < \infty$ iff [because $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$ exists] $\frac{1}{f'} \in \Theta\left(\frac{1}{g'}\right)$. This completes the proof of the second equivalence. This completes the proof. $\square$

Below, several examples of $T(n)$ and corresponding $Max(t)$ and the derivative $Max'(t)$ are described. For the cases 8 through 13 the reciprocals $\frac{1}{Max'(t)}$ are included. All cases 1 through 13 may be considered benchmark cases.

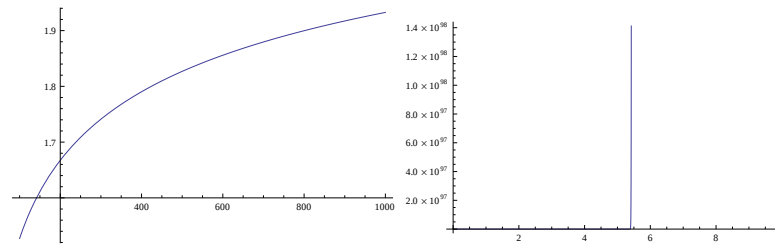
Particularly important are cases: 1, 3, 7, 8, 9, 11, and 12.

Note different scales used in graphs of sample functions below.

1.

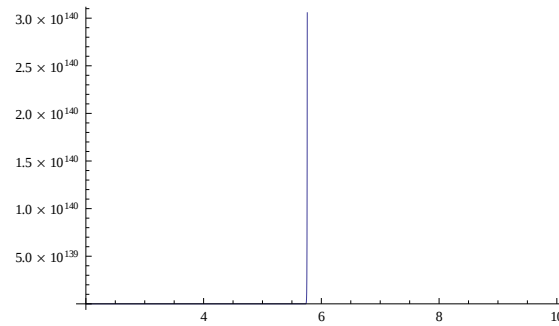
$$T(n) \in \Theta(\log \log n) \dots \text{Max}(t) \in \Theta(a^{b^t}); \text{ for some } a, b > 1$$

Here are graphs of $\ln \ln n$ and e^{e^t} :



$$\text{Max}'(t) \in \Theta(a^{b^t} b^t)$$

Here is a graph of $e^{e^t} e^t = e^{e^t+t}$:

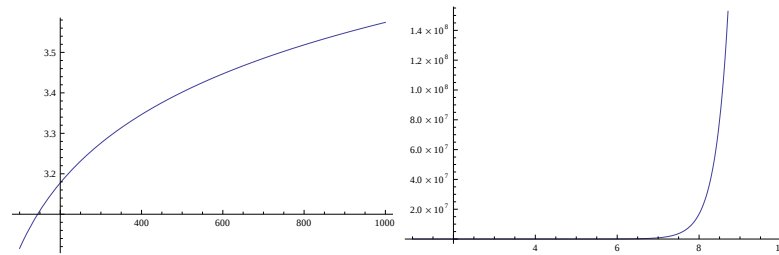


In this case, the larger t the (dramatically) more it pays off to run P on a faster computer.

2.

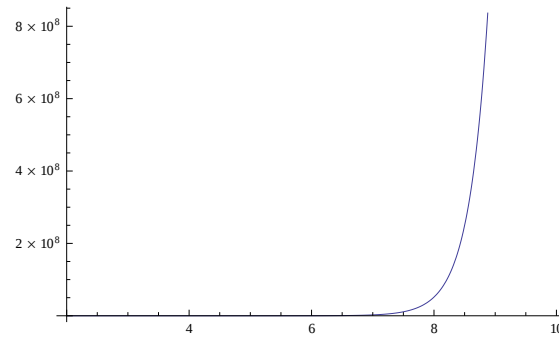
$$T(n) \in \Theta\left(\frac{\log n}{\log \log n}\right) \dots \text{Max}(t) \in \Omega(at)^{at} \cap O((bt)^{bt}); \text{ for some } a, b > 1$$

Here are graphs of $\frac{\log n}{\log \log n}$ and t^t :



$$\text{Max}'(t) \in \Omega((at)^{at} \ln t) \cap O((bt)^{bt} \ln t)$$

Here is a graph of $t^t \ln t$:

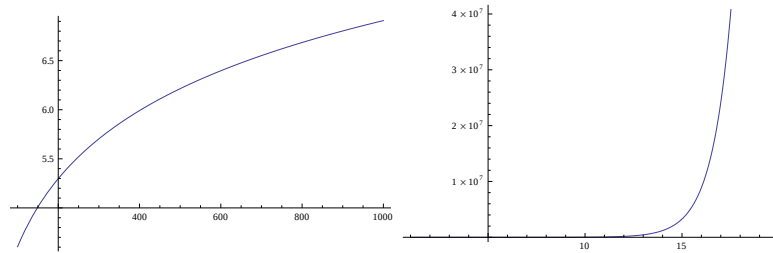


In this case, the larger t the (significantly) more it pays off to run P on a faster computer.

3.

$$T(n) \in \Theta(\log n) \dots Max(t) \in \Theta(a^t); \text{ for some } a > 1$$

Here are graphs of $\ln n$ and e^t :



$$Max'(t) \in \Theta(a^t)$$

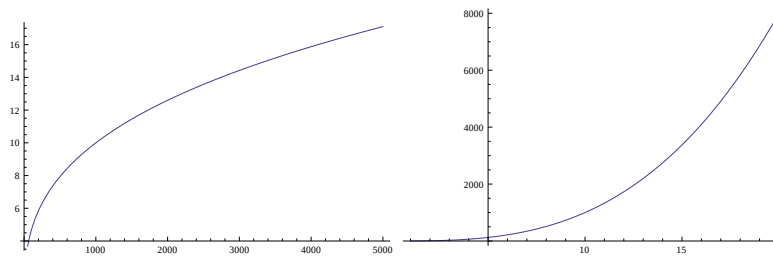
See above for a graph of e^t .

In this case, the larger t the (significantly) more it pays off to run P on a faster computer.

4.

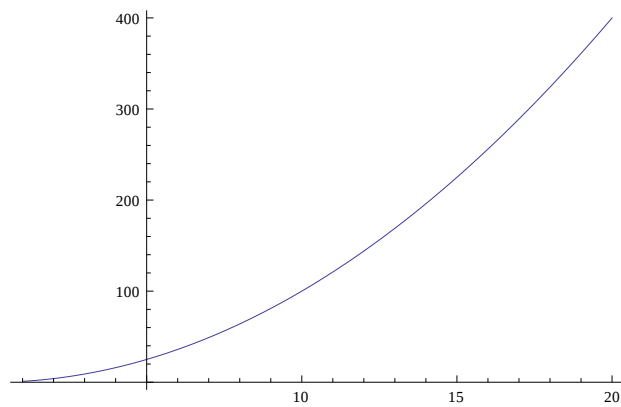
$$T(n) \in \Theta(\sqrt[3]{n}) \dots Max(t) \in \Theta(t^3)$$

Here are graphs of $\sqrt[3]{n}$ and t^3 :



$$Max'(t) \in \Theta(t^2)$$

Here is a graph of t^2 :

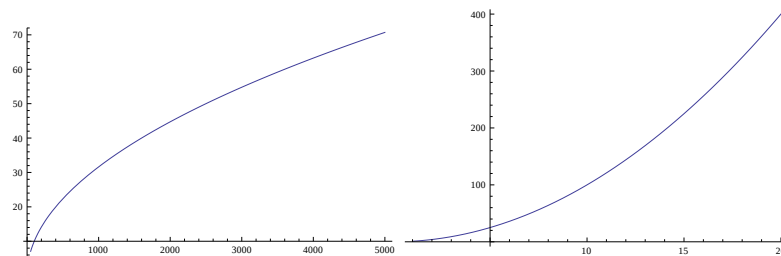


In this case, the larger t the more it pays off to run P on a faster computer.

5.

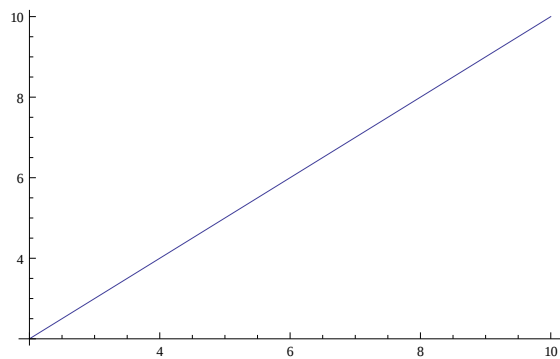
$$T(n) \in \Theta(\sqrt{n}) \dots Max(t) \in \Theta(t^2)$$

Here are graphs of $\sqrt{n}$ and t^2 :



$$Max'(t) \in \Theta(t)$$

Here is a graph of t :

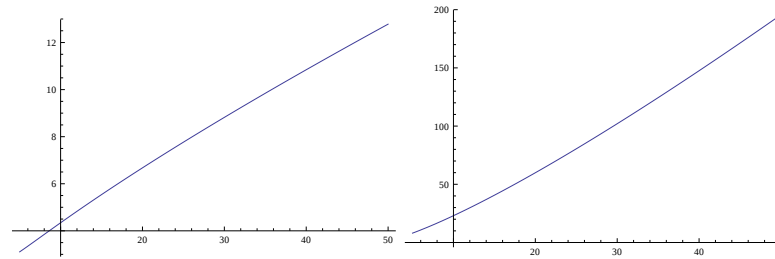


In this case, the larger t the more it pays off to run P on a faster computer.

6.

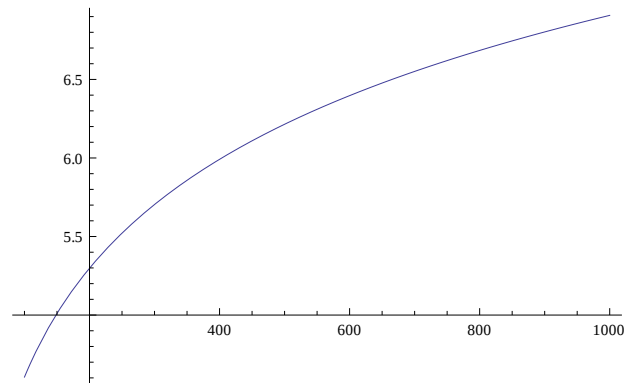
$$T(n) \in \Theta\left(\frac{n}{\log n}\right) \dots \text{Max}(t) \in \Theta(t \log t)$$

Here are graphs of $\frac{n}{\log n}$ and $t \ln t$:



$$\text{Max}'(t) \in \Theta(\log t)$$

Here is a graph of $\ln n$:

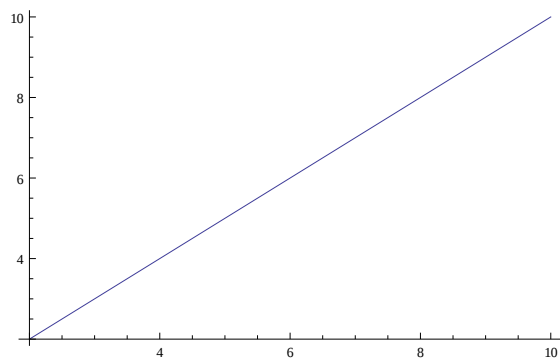


In this case, the larger t the (moderately) more it pays off to run P on a faster computer.

7.

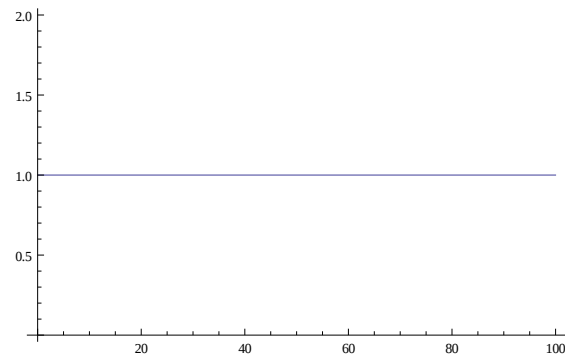
$$T(n) \in \Theta(n) \dots Max(t) \in \Theta(t)$$

Here is a graphs of n and t :



$$Max'(t) \in \Theta(1)$$

Here is a graph of 1:

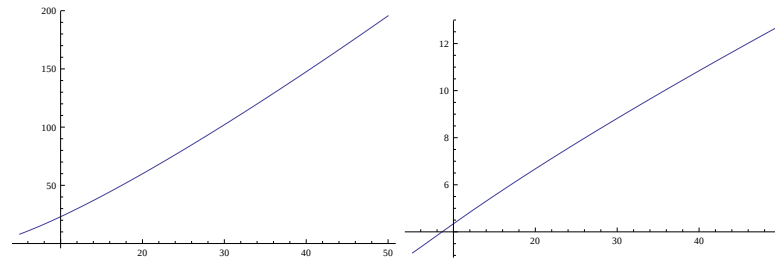


In this case, the increase of the maximum size of input in function of speed of the computer is constant for all t , so the payoff for running P on a faster computer remains roughly the same for all sizes of its inputs.

8.

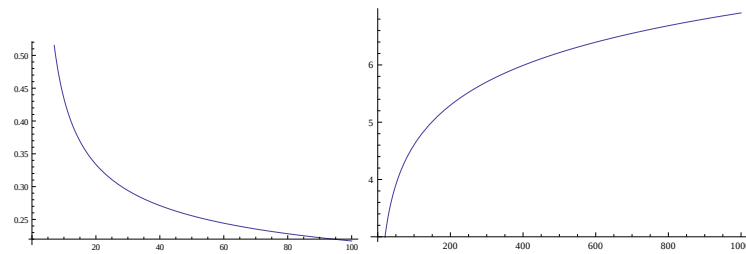
$$T(n) \in \Theta(n \log n) \dots \text{Max}(t) \in \Theta\left(\frac{t}{\log t}\right)$$

Here are graphs of $n \log n$ and $\frac{t}{\log t}$:



$$Max'(t) \in \Theta(\frac{1}{\log t}); \frac{1}{Max'(t)} \in \Theta(\log t)$$

Here are graphs of $\frac{1}{\log t}$ and $\log t$:

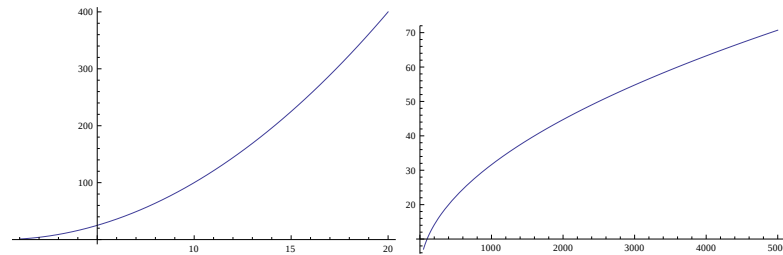


In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (slightly) more does it cost to accomplish the unit increase of the tractable input to P .

9.

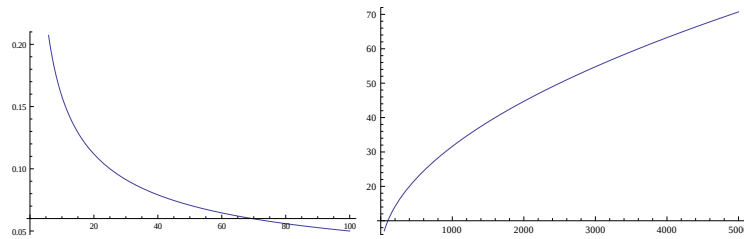
$$T(n) \in \Theta(n^2) \dots Max(t) \in \Theta(\sqrt{t})$$

Here are graphs of n^2 and $\sqrt{t}$:



$$Max'(t) \in \Theta(\frac{1}{\sqrt{t}}); \frac{1}{Max'(t)} \in \Theta(\sqrt{t})$$

Here are graphs of $\frac{1}{\sqrt{t}}$ and $\sqrt{t}$:

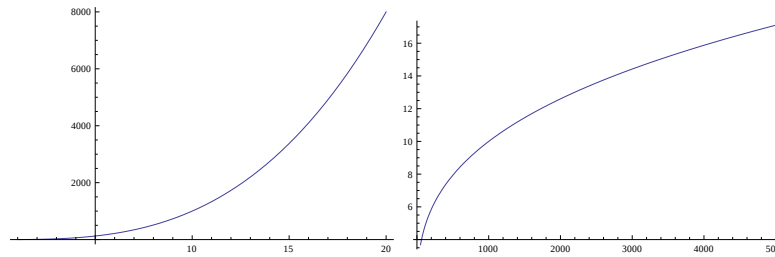


In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (moderately) more does it cost to accomplish the unit increase of the tractable input to P .

10.

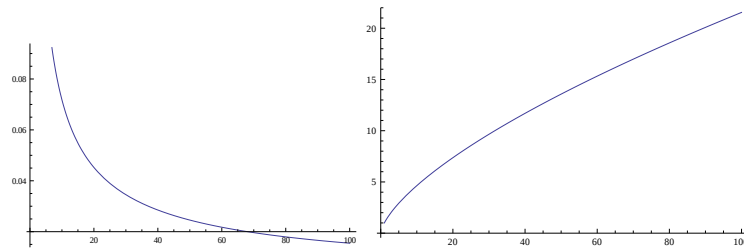
$$T(n) \in \Theta(n^3) \dots Max(t) \in \Theta(\sqrt[3]{t})$$

Here are graphs of n^3 and $\sqrt[3]{t}$:



$$Max'(t) \in \Theta(\frac{1}{\sqrt[3]{t^2}}); \frac{1}{Max'(t)} \in \Theta(\sqrt[3]{t^2})$$

Here are graphs of $\frac{1}{\sqrt[3]{t^2}}$ and $\sqrt[3]{t^2}$:

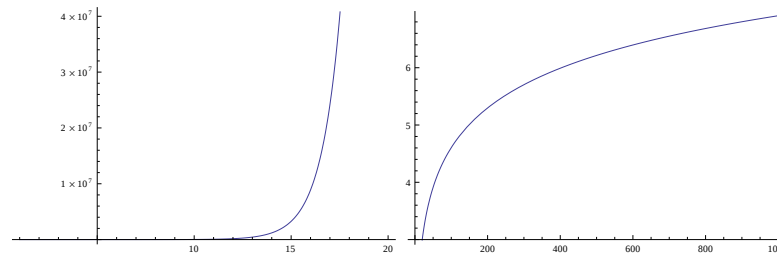


In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (moderately) more does it cost to accomplish the unit increase of the tractable input to P .

11.

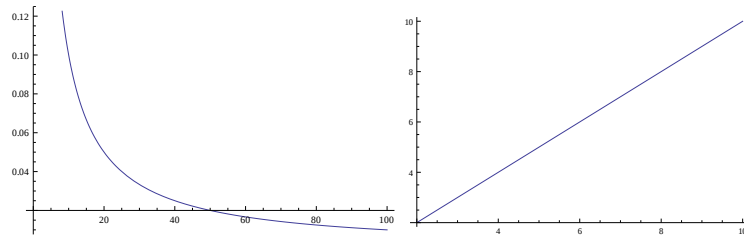
$$T(n) \in \Theta(a^n) \dots Max(t) \in \Theta(\log t); \text{ for all } a > 1$$

Here are graphs of e^n and $\ln t$:



$$Max'(t) \in \Theta(\frac{1}{t}); \frac{1}{Max'(t)} \in \Theta(t)$$

Here are graphs of $\frac{1}{t}$ and t :

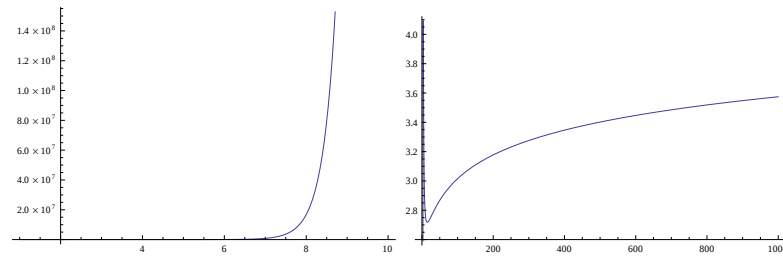


In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P .

12.

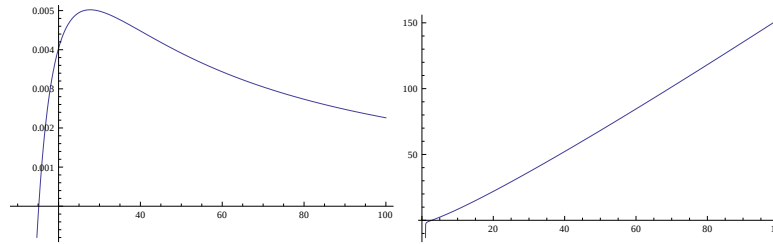
$$T(n) \in \Theta((an)^{bn}) \dots Max(t) \in \Theta(\frac{\log t}{\log \log t}); \text{ for all } a, b > 1$$

Here are graphs of n^n and $\frac{\log t}{\log \log t}$:



$$Max'(t) \in \Theta(\frac{1}{t \ln \ln t} - \frac{1}{t(\ln \ln t)^2}) = \Theta(\frac{1}{t \ln \ln t}); \quad \frac{1}{Max'(t)} \in \Theta(t \ln \ln t)$$

Here are graphs of $\frac{1}{t \ln \ln t} - \frac{1}{t(\ln \ln t)^2}$ and $t \ln \ln t$:

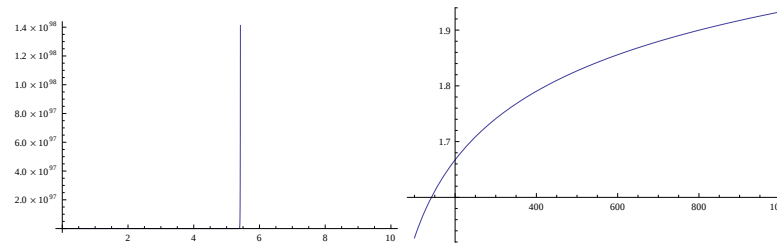


In this case, the larger t the (dramatically) less it pays off to run P on a faster computer. More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P .

13.

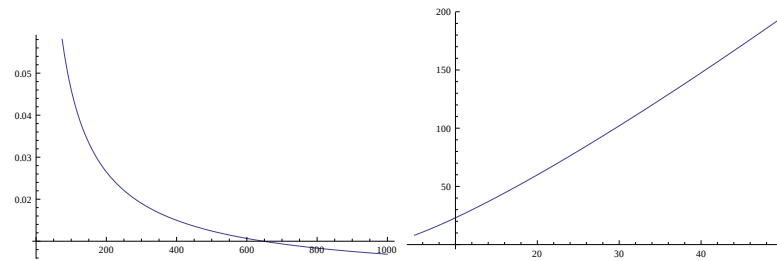
$$T(n) \in \Theta(a^{b^n}) \dots Max(t) \in \Theta(\log \log t); \text{ for all } a, b > 1$$

Here are graphs of e^{e^n} and $\log \log t$:



$$Max'(t) \in \Theta(\frac{1}{t \ln t}); \frac{1}{Max'(t)} \in \Theta(t \ln t)$$

Here are graphs of $\frac{1}{t \ln t}$ and $t \ln t$:



In this case, the larger t the (dramatically) less it pays off to run P on a faster computer. More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P .

1.1 Proofs (optional for all students)

1. The inverse function of the function

$$n = a^{b^t}$$

is

$$t = \log_b \log_a n.$$

Since for every $a, b > 0$,

$$\lim_{n \rightarrow \infty} \frac{\log_b \log_a n}{\ln \ln n}$$

exists and is between 0 and ∞ ,

$$\Theta(\log_b \log_a n) = \Theta(\log \log n).$$

2. The inverse function of the function

$$n = (at)^{at}$$

is

$$t = \frac{\ln n}{aW(\ln n)}$$

where $W(x)$ is the Lambert's W function approximated by:

$$W(x) = \ln x - \ln \ln x \pm O\left(\frac{\ln \ln x}{\ln x}\right).$$

Therefore, the inverse function of the function

$$n = (at)^{at}$$

is in

$$\Theta\left(\frac{\ln n}{\ln \ln n - \ln \ln \ln n}\right).$$

Since

$$\lim_{n \rightarrow \infty} \frac{\frac{\ln n}{\ln \ln n - \ln \ln \ln n}}{\frac{\ln n}{\ln \ln n}}$$

exists and is between 0 and ∞ ,

$$\Theta\left(\frac{\ln n}{\ln \ln n - \ln \ln \ln n}\right) = \Theta\left(\frac{\ln n}{\ln \ln n}\right).$$

For $0 < a < b$ we have $(at)^{at} < (bt)^{at} < (at)^{bt} < (bt)^{bt}$. Hence, for each increasing function $f \in \Theta\left(\frac{\ln n}{\ln \ln n}\right)$, and some $a > 0$, $f^{-1} \in \Omega((at)^{at})$. Similarly, for some $b > 0$, $f^{-1} \in O((bt)^{bt})$.

3. The inverse function of the function

$$n = a^t$$

is

$$t = \log_a n.$$

Since for every $a > 0$,

$$\lim_{n \rightarrow \infty} \frac{\log_a n}{\ln n}$$

exists (it's $\frac{1}{\ln a}$) and is between 0 and ∞ ,

$$\Theta(\log_a n) = \Theta(\log n).$$

4. Left as an exercise.

5. Left as an exercise.
6. The inverse function of the function

$$n = t \log_a t$$

is

$$t = \frac{n \ln a \ln n}{W(n \ln a \ln n)}$$

where $W(x)$ is the Lambert's W function approximated by:

$$W(x) = \ln x - \ln \ln x \pm O\left(\frac{\ln \ln x}{\ln x}\right).$$

Therefore, the inverse function of the function

$$t = n \log_a n$$

is in

$$\Theta\left(\frac{n \ln a \ln n}{\ln(n \ln a \ln n) - \ln \ln(n \ln a \ln n)}\right).$$

Since

$$\lim_{n \rightarrow \infty} \frac{\frac{n \ln a \ln n}{\ln(n \ln a \ln n) - \ln \ln(n \ln a \ln n)}}{\frac{n}{\ln n}}$$

exists and is between 0 and ∞ ,

$$\Theta\left(\frac{n \ln a \ln n}{\ln(n \ln a \ln n) - \ln \ln(n \ln a \ln n)}\right) = \Theta\left(\frac{n}{\ln n}\right).$$

7. Left as an exercise.
8. The inverse function of the function

$$t = n \log_a n$$

is

$$n = \frac{t \ln a \ln t}{W(t \ln a \ln t)}$$

where $W(x)$ is the Lambert's W function approximated by:

$$W(x) = \ln x - \ln \ln x \pm O\left(\frac{\ln \ln x}{\ln x}\right).$$

Therefore, the inverse function of the function

$$n = t \log_a t$$

is in

$$\Theta\left(\frac{t \ln a \ln t}{\ln(t \ln a \ln t) - \ln \ln(t \ln a \ln t)}\right).$$

Since

$$\lim_{t \rightarrow \infty} \frac{\frac{t \ln a \ln t}{\ln(t \ln a \ln t) - \ln \ln(t \ln a \ln t)}}{\frac{t}{\ln t}}$$

exists and is between 0 and ∞ ,

$$\Theta\left(\frac{t \ln a \ln t}{\ln(t \ln a \ln t) - \ln \ln(t \ln a \ln t)}\right) = \Theta\left(\frac{t}{\ln t}\right).$$

9. Left as an exercise.
10. Left as an exercise.
11. The inverse function of the function

$$t = a^n$$

is

$$n = \log_a t.$$

Since for every $a > 0$,

$$\lim_{t \rightarrow \infty} \frac{\log_a t}{\ln t}$$

exists (it's $\frac{1}{\ln a}$) and is between 0 and ∞ ,

$$\Theta(\log_a t) = \Theta(\log t).$$

12. The inverse function of the function

$$t = (an)^{an}$$

is

$$n = \frac{\ln t}{aW(\ln t)}$$

where $W(x)$ is the Lambert's W function approximated by:

$$W(x) = \ln x - \ln \ln x \pm O\left(\frac{\ln \ln x}{\ln x}\right).$$

Therefore, the inverse function of the function

$$t = (an)^{an}$$

is in

$$\Theta\left(\frac{\ln t}{\ln \ln t - \ln \ln \ln t}\right).$$

Since

$$\lim_{t \rightarrow \infty} \frac{\frac{\ln t}{\ln \ln t - \ln \ln \ln t}}{\frac{\ln t}{\ln \ln t}}$$

exists and is between 0 and ∞ ,

$$\Theta\left(\frac{\ln t}{\ln \ln t - \ln \ln \ln t}\right) = \Theta\left(\frac{\ln t}{\ln \ln t}\right).$$

For $0 < a < b$ we have $(an)^{an} < (bn)^{an} < (an)^{bn} < (bn)^{bn}$. Hence, for each increasing function $f \in \Theta(\frac{\ln n}{\ln \ln n})$, and some $a > 0$, $f^{-1} \in \Omega((an)^{an})$. Similarly, for some $b > 0$, $f^{-1} \in O((bn)^{bn})$.

13. The inverse function of the function

$$t = a^{b^n}$$

is

$$n = \log_b \log_a t.$$

Since for every $a, b > 0$,

$$\lim_{t \rightarrow \infty} \frac{\log_b \log_a t}{\ln \ln t}$$

exists and is between 0 and ∞ ,

$$\Theta(\log_b \log_a t) = \Theta(\log \log t).$$