CSC 501/401 Analysis of Algorithms Spring '15

Practice Exam

ANSWERS

PART I: MULTIPLE-CHOICE QUESTIONS

1. Give a closed-form formula for the following long summation:

$$\sum_{i=1}^{n} i \times 2^i.$$

   (A) $n \times 2^{n+1} - 1$
   (B) $(n-1)2^{n+1} + 2$ **CORRECT**
   (C) $(n+1)2^{\lfloor \lg n \rfloor + 1} + 2$
   (D) $\lceil \lg(n+1) \rceil$

2. Same as above.

3. What is the least power of 2 greater than n?

   (A) $\lceil x \rceil$, where $n \leq x < n+1$
   (B) $2^{\lceil x \rceil}$, where $n \leq x < n+1$
   (C) $2^{\lfloor \log_2 n \rfloor + 1}$ **CORRECT**

(D) $2^{\lceil \log_2 n \rceil - 1}$

4. Same as above.

5. Which expression defines the *conditional expected value* (a.k.a. *conditional expectation*) $E(f|S)$ of random variable f under condition S?

   (A) $E(f) \times Pr(S)$
   (B) $\sum_{e \in S} f(e) \times Pr(e)$
   (C) $\sum_{e \in S} f(e) \times Pr(e|S)$ **CORRECT**
   (D) $\sum_{e \in U} f(e) \times Pr(e)$

   where U is the set of all elementary events and $Pr(x)$ is probability of x.

6. Same as above.

7. Which of the following inequalities is true for every non-increasing integrable function f, assuming $a + 1 < b$?

   (A) $\int_{a-1}^{b} f(x)dx \geq \sum_{n=a}^{b} f(n) \geq \int_{a}^{b+1} f(x)dx$ **CORRECT**
   (B) $\int_{a}^{b-1} f(x)dx \geq \sum_{n=a}^{b} f(n) \geq \int_{a+1}^{b} f(x)dx$
   (C) $\int_{a}^{b} f(x)dx \geq \sum_{n=a}^{b} f(n) \geq \int_{a+1}^{b+1} f(x)dx$
   (D) $\int_{a-1}^{b-1} f(x)dx \geq \sum_{n=a}^{b} f(n) \geq \int_{a}^{b} f(x)dx$

8. Same as above.

9. Let $f(n) = (n \log_2 n)^{1.5}$ and $g(n) = n^{1.56} + 0.5\sqrt{\log_2 n}$. Which of the following is true?

   (A) $g \in O(f)$
   (B) $f \in \Theta(g)$
   (C) $g \in o(f)$
   (D) $f \in o(g)$ **CORRECT**

10. Same as above.

11. What is the worst-case number of comparisons of keys performed by unsuccessful sequential search on an n-element ordered array?

(A) n **CORRECT**

(B) $\frac{n+1}{2}$

(C) $\lfloor\frac{n+1}{2}\rfloor$

(D) $\frac{n}{n+1} + \frac{n}{2}$

12. Same as above.

13. What is the average-case number of comparisons of keys performed by successful binary search on an n-element ordered array?

(A) $\lfloor\frac{n}{2}\rfloor$

(B) $\lfloor\log_2 n\rfloor + 1$

(C) $\log_2(n+1) - 1 + o(1)$ **CORRECT**

(D) $n - 1$

14. Same as above.

15. What is the average number of comparisons of keys that `InsertionSort` performs while sorting an n-element array?

(A) $n^2 \log n$

(B) $\frac{n(n+1)}{4}$

(C) $\frac{(n-1)(n+2)}{4}$

(D) $\frac{n(n-1)}{4} + \sum_{i=1}^{n-1} \frac{i}{i+1}$ **CORRECT**

16. Same as above.

17. Let C be a class of sorting algorithms that remove at most one inversion after each comparison. Which of the following is a lower bound on the average number of comparisons of keys that every algorithm on C must perform while sorting an n-element array?

(A) $n^2 \log n$

(B) $\frac{n(n+1)}{4}$

(C) $\frac{(n-1)(n+2)}{4}$ **CORRECT**

(D) $\frac{n(n-1)}{4} + \sum_{i=1}^{n-1} \frac{i}{i+1}$

18. Same as above.

19. What is the shortest *ipl* (the *internal path length*) in any binary tree on n nodes?

    (A) $ipl = \sum_{i=1}^{n} \lfloor \lg i \rfloor$
    (B) $ipl = (n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2$
    (C) $ipl = (n+1)(\lg(n+1) + \varepsilon(n+1)) - 2n$, where $\varepsilon(n) = 1 + \theta - 2^{\theta}$ and $\theta = \lceil \lg\, n \rceil - \lg\, n$.
    (D) All the above. **CORRECT**

20. Same as above.

21. How many comparisons of keys are performed in the worst case during the execution of `Mergesort` on an n-element array?

    (A) $n(\lceil \log_2 n \rceil + 1) - 2^{\lceil \log_2 n \rceil}$
    (B) $n(\lfloor \log_2 (n+1) \rfloor + 1) - 2^{\lfloor \log_2 (n+1) \rfloor}$
    (C) $n\lceil \log_2 (n+1) \rceil - 2^{\lceil \log_2 (n+1) \rceil} + 1$ **CORRECT**
    (D) $n\lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor} + 1$

22. Same as above.

PART II: ESSAY-TYPE QUESTIONS

23. Prove by induction the following equality:

$$\sum_{i=1}^{n}(2i-1) = n^2$$

Answer (Based on Chapter 3, Section 3.4).

Basis step ($n = 1$)

$$\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{1}(2i-1) = 2 \times 1 - 1 = 1^2 = n^2$$

Inductive step

$$\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1 =$$

(by inductive hypothesis)

$$= n^2 + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2$$

24. Prove that for every integer $n \geq 1$,

$$\lfloor \log_2 n \rfloor + 1 = \lceil \log_2(n+1) \rceil.$$

Answer

$$\lfloor \text{Log2}[n] \rfloor \leq \text{Log2}[n] < \lfloor \text{Log2}[n] \rfloor + 1$$

$$2^{\lfloor \text{Log2}[n] \rfloor} \leq 2^{\text{Log2}[n]} < 2^{\lfloor \text{Log2}[n] \rfloor + 1}$$

$$2^{\lfloor \text{Log2}[n] \rfloor} \leq n < 2^{\lfloor \text{Log2}[n] \rfloor + 1}$$

$$2^{\lfloor \text{Log2}[n] \rfloor} < n + 1 \leq 2^{\lfloor \text{Log2}[n] \rfloor + 1}$$

$$\text{Log2}\left[2^{\lfloor \text{Log2}[n] \rfloor}\right] < \text{Log2}[n+1] \leq \text{Log2}\left[2^{\lfloor \text{Log2}[n] \rfloor + 1}\right]$$

$$\lfloor \text{Log2}[n] \rfloor < \text{Log2}[n+1] \leq \lfloor \text{Log2}[n] \rfloor + 1$$

$$\lfloor \text{Log2}[n] \rfloor < \lceil \text{Log2}[n+1] \rceil \leq \lfloor \text{Log2}[n] \rfloor + 1$$

$$\lceil \text{Log2}[n+1] \rceil = \lfloor \text{Log2}[n] \rfloor + 1$$

25. Find a best-case array E with 15 distinct elements for Quicksort. How many comparisons did QuickSort made while sorting that array?

Answer

$$E = \{8, 4, 12, 2, 6, 10, 14, 1, 3, 5, 7, 9, 11, 13, 15\}$$

The number of comparisons is:

$$B(15) = \sum_{i=1}^{15} \lfloor \lg i \rfloor.$$

Using Mathematica, one gets:

```
In[18]:=
∑_{i=1}^{15} ⌊Log2[i]⌋
Out[18]=
34
```

26. **Graduate Credit** Prove that for any integer $b, c \geq 2$, if

$$f(n) \in \Omega(n^{\log_c b + \epsilon})$$

for some positive ϵ, and

$$f(n) \in O(n^{\log_c b + \delta})$$

for some $\delta > \epsilon$, then any solution of the recurrence equation

$$T(n) = bT(\frac{n}{c}) + f(n)$$

satisfies

$$T(n) \in \Theta(f(n)).$$

Answer See proof of Theorem 3.17 case 3 in your textbook, Chapter 3 Section 3.7.1. page 139.