CSC 501/401

Lectures on

Analysis of Algorithms

by

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Computer Science
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Excerpts from

Sums of floors and ceilings of consecutive logarithms

http://csc.csudh.edu/suchenek/CSC401/Slides/Knuth-Suchenek_formulas_sums_of_floors_ceilings_logs.pdf

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Theorem 1.4 *For every* $x > 0$,

$$x\lceil \lg x \rceil - 2^{\lceil \lg x \rceil} = x(\lg x + \varepsilon(x) - 1),$$

$$\varepsilon(x) = 1 + \lceil \lg x \rceil - \lg x - 2^{\lceil \lg x \rceil - \lg x},$$

$$\lg x + \varepsilon(x) - 1 = \lceil \lg x \rceil - \frac{2^{\lceil \lg x \rceil}}{2^{\lg x}},$$

$$\lg x + \varepsilon(x) - 1 = \lceil \lg x \rceil - \frac{2^{\lceil \lg x \rceil}}{x},$$

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$$x(\lg\, x + \varepsilon(x) - 1) = x\lceil \lg\, x \rceil - 2^{\lceil \lg\, x \rceil},$$

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Theorem 2.1 *For every natural number* $n \geq 1$,

$$\sum_{i=1}^{n} \lceil \lg i \rceil = n \lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1.$$

$$\sum_{i=1}^{n} \lceil \lg i \rceil = n\lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1 =$$

$$= n\lceil \lg(n+1) \rceil - 2^{\lceil \lg(n+1) \rceil} + 1 =$$

$$= n\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + n + 1 =$$

$$= n(\lg n + \varepsilon(n)) - n + 1.$$

$$\sum_{i=1}^{n} \lceil \lg i \rceil = n \lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1 =$$

$$= n \lceil \lg (n+1) \rceil - 2^{\lceil \lg (n+1) \rceil} + 1 =$$

$$= n \lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + n + 1 =$$

$$= n(\lg n + \varepsilon(n)) - n + 1.$$

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Theorem 1.3 *For every* $x > 0$,

$$x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1} = x(\lg x + \alpha(x) - 2),$$

$$\alpha(x) = 2 - (\lg x - \lfloor \lg x \rfloor) - 2^{1-(\lg x - \lfloor \lg x \rfloor)},$$

$$\alpha(x) = 2 - \lg x + \lfloor \lg x \rfloor - \frac{2^{\lfloor \lg x \rfloor + 1}}{2^{\lg x}},$$

$$\lg x + \alpha(x) - 2 = \lfloor \lg x \rfloor - \frac{2^{\lfloor \lg x \rfloor + 1}}{x},$$

$$x(\lg x + \alpha(x) - 2) = x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1}.$$

$$\alpha(x) = 2 - (\lg\, x - \lfloor \lg\, x \rfloor) - 2^{1-(\lg\, x - \lfloor \lg\, x \rfloor)},$$

$$\alpha(x) = 2 - \lg\, x + \lfloor \lg\, x \rfloor - \frac{2^{\lfloor \lg\, x \rfloor + 1}}{2^{\lg\, x}},$$

$$\lg\, x + \alpha(x) - 2 = \lfloor \lg\, x \rfloor - \frac{2^{\lfloor \lg\, x \rfloor + 1}}{x},$$

$$x(\lg\, x + \alpha(x) - 2) = x\lfloor \lg\, x \rfloor - 2^{\lfloor \lg\, x \rfloor + 1}.$$

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$$\alpha(x) = 2 - \lg\ x + \lfloor \lg\ x \rfloor - \frac{2^{\lfloor \lg\ x \rfloor + 1}}{2^{\lg\ x}},$$

$$\lg\ x + \alpha(x) - 2 = \lfloor \lg\ x \rfloor - \frac{2^{\lfloor \lg\ x \rfloor + 1}}{x},$$

$$x(\lg\ x + \alpha(x) - 2) = x\lfloor \lg\ x \rfloor - 2^{\lfloor \lg\ x \rfloor + 1}.$$

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$$x(\lg x + \alpha(x) - 2) = x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1}.$$

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Theorem 1.5 *For every x,*

$$\varepsilon(x) = \alpha(x),$$

where ε is given by:

$$\varepsilon(n) = 1 + \theta - 2^{\theta} \text{ and } \theta = \lceil \lg x \rceil - \lg x,$$

and α is given by:

$$\alpha(x) = 2 - \varphi - 2^{1-\varphi} \text{ and } \varphi = \lg x - \lfloor \lg x \rfloor.$$

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Corollary 1.6 *For every* $x > 0$,

$$x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1} = x(\lg x + \varepsilon(x) - 2),$$

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2 =$$

$$= \boxed{(n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1}} + 2 =$$

$$= (n+1)(\lg(n+1) + \varepsilon(n+1)) - 2n,$$

Put x = n+1

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2 =$$

$$= (n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2 =$$

$$= (n+1)(\lg(n+1) + \varepsilon(n+1)) - 2n,$$

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2 =$$

$$= (n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2 =$$

$$= (n+1)(\lg(n+1) + \varepsilon(n+1)) - 2n,$$

Minimal internal path length $ipl_{min}(n)$

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2 =$$

$$= (n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2 =$$

$$= (n+1)(\lg(n+1) + \varepsilon(n+1)) - 2n,$$

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2 =$$

$$= (n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2 =$$

$$\boxed{= (n+1)(\lg(n+1) + \varepsilon(n+1))} - 2n,$$

Minimal external path length $epl_{min}(n)$

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