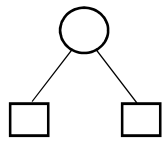
Theorem For every 2-tree $T_n$ with n internal nodes,

$$epl\ (T_n) = ipl\ (T_n) + 2n$$

Proof by induction on n.

Basic step (n = 1)



$ipl\ (T_1) = 0$

$epl\ (T_1) = 1 + 1 = 2$

So, $epl\ (T_1) = ipl\ (T_1) + 2 \times 1$

This completes the basic step.

Inductive step

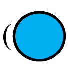
Inductive hypothesis assume that for sum $n \geq 1$, and for every 2-tree $T_n$ with n internal nodes,

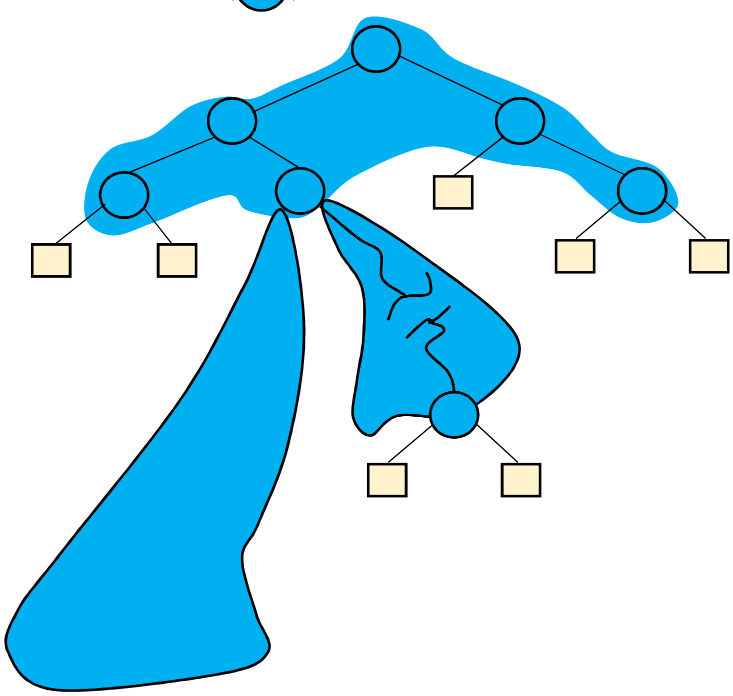
$$epl\ (T_n) = ipl\ (T_n) + 2n$$

We are going to prove that for every 2-tree $T_{n+1}$ with $n + 1$ internal nodes,

$$epl\ (T_{n+1}) = ipl\ (T_{n+1}) + 2(n + 1).$$

Let $T_{n+1}$ be any 2-tree with $n + 1$ internal nodes.

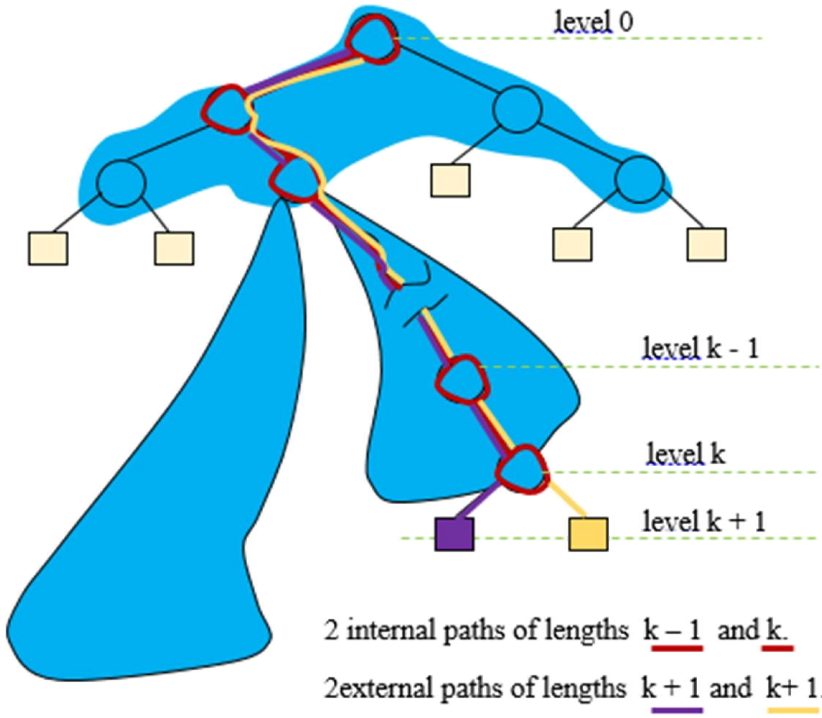
2-tree $T_{n+1}$ with $n + 1$ internal nodes ()



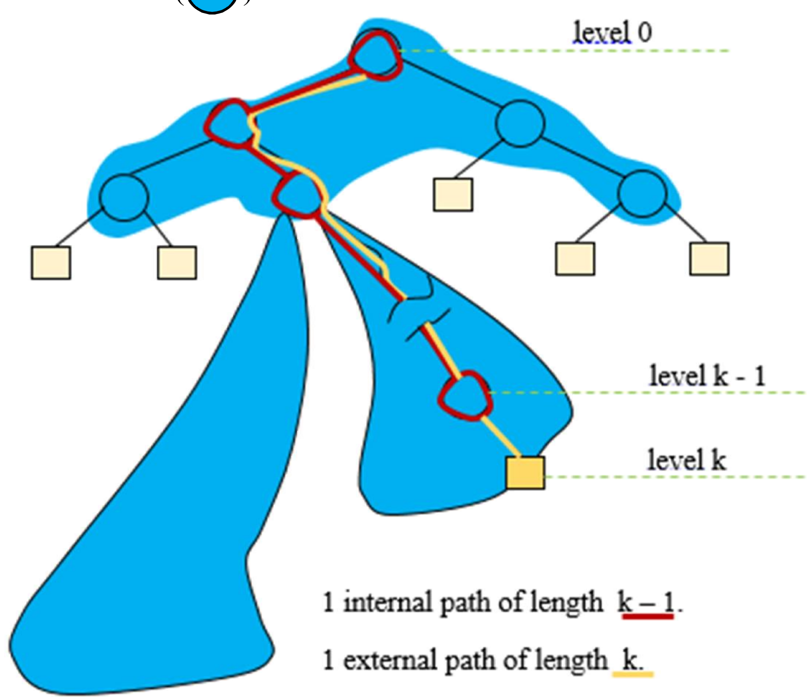
Let's remove one internal leaf from $T_{n+1}$. (It must have an internal leaf befause it is non-empty and finite.) The resulting 2-tree $T_n$ has n internal leaves, so the inductive hypothesis does apply!

$$\text{epl}(T_n) = \text{ipl}(T_n) + 2n$$

2-tree $T_{n+1}$ with n+1 internal nodes ( )




2-tree Tn with n internal nodes ( )



The differences are:

$$\text{epl}(T_{n+1}) - \text{epl}(T_n)$$
$$= k + 1 + k + 1 - k$$
$$= k + 2$$

$$\text{ipl}(T_{n+1}) - \text{ipl}(T_n)$$
$$= k - 1 + k - (k - 1) = k$$

Now,

$$\text{epl}(T_{n+1}) = \text{epl}(T_n) + k + 2$$
$$= \text{(by the inductive hypothesis)}$$
$$\text{ipl}(T_n) + 2n + k + 2$$
$$= \text{ipl}(T_{n+1}) + 2(n + 1).$$

This completes the inductive step which completes the proof.