Sums of floors and ceilings of consecutive logarithms

For in-class use only in CSC 501/401 course

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Abstract

For every natural number $n \geq 1$,

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2 =$$

$$= (n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2 =$$

$$= (n+1)(\lg(n+1) + \varepsilon(n+1)) - 2n,$$

where ε, given by:

$$\varepsilon(n) = 1 + \theta - 2^{\theta} \text{ and } \theta = \lceil \lg\, n \rceil - \lg\, n,$$

is a continuous function of n on the set of reals > 1, with the minimum value 0 and and the maximum (*supremum*, if n is restricted to integers) value

$$\delta = 1 - \lg e + \lg \lg e \approx 0.0860713320559342.$$

Hence,

$$(n+1)\lg(n+1) - 2n \leq \sum_{i=1}^{n} \lfloor \lg i \rfloor \leq (n+1)\lg(n+1) - 1.913n + 0.0861.$$

For every natural number $n \geq 1$,

$$\sum_{i=1}^{n} \lceil \lg i \rceil = n \lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1 =$$

$$= n \lceil \lg (n+1) \rceil - 2^{\lceil \lg (n+1) \rceil} + 1 =$$

$$= n \lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + n + 1 =$$

$$= n(\lg n + \varepsilon(n)) - n + 1.$$

Hence

$$n \lg n - n + 1 \leq \sum_{i=1}^{n} \lceil \lg i \rceil \leq n \lg n - 0.913n + 1.$$

Moreover, for every natural number $n \geq 1$,

$$\sum_{i=1}^{n} (\lceil \lg i \rceil - \lfloor \lg i \rfloor) = n - \lfloor \lg n \rfloor - 1.$$

1 A sum of floors of consecutive logarithms

Theorem 1.1 *For every natural number $n \geq 1$,*

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2. \quad (1)$$

Note. The right-hand side of (1) is not a continuous function.

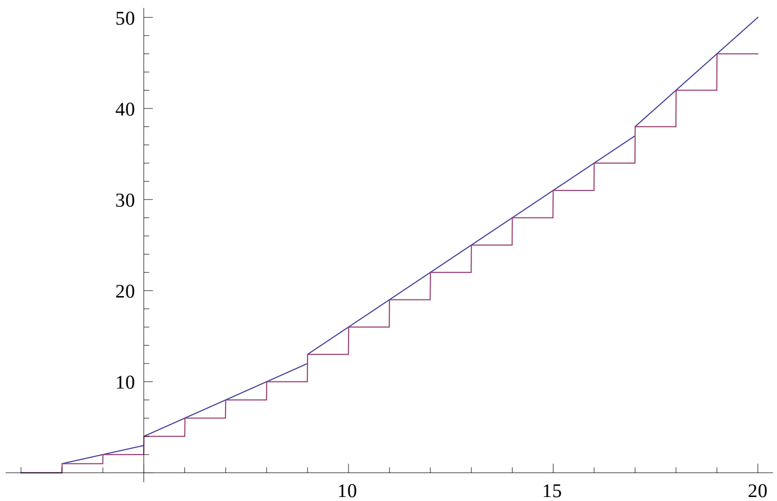

Figure 1: Functions $\sum_{i=1}^{n} \lfloor \lg i \rfloor$ and $(n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2$.

Proof in the file (easy, by solving equation $x + y = n + 1; x + \frac{y}{2} = 2^{\lfloor \lg n \rfloor}$)
/media/Suchenek/Courses/CSC401/Slides/2-trees.htm
and in the file (by direct calculation with Rieman's sum method)
/media/Suchenek/Courses/CSC311/Materials_for_text/Balanced_tree.pdf
. □

Theorem 1.2 *For every natural number $n \geq 1$,*

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2. \tag{2}$$

Note. The right-hand side of (2) is a continuous function.

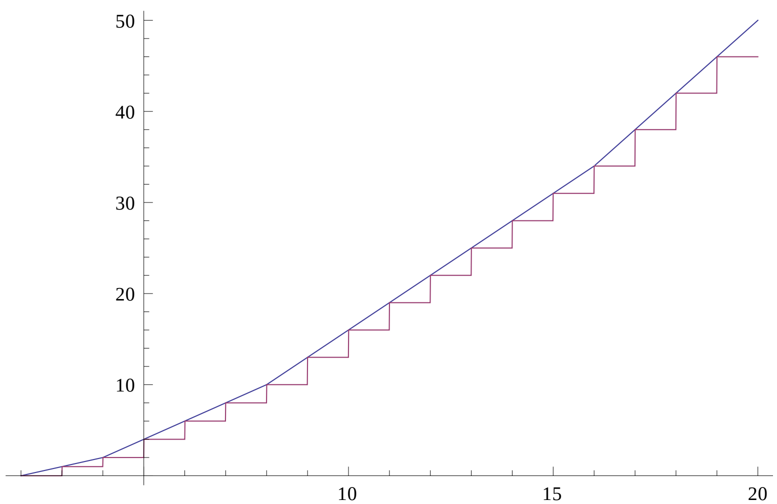

Figure 2: Functions $\sum_{i=1}^{n} \lfloor \lg i \rfloor$ and $(n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2$.

Note. Function $f(n) = \sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2$ is a linear interpolation of itself restricted to $n = 2^k$. In particular, it is a linear interpolation of the function $g(k) = k(2^k + 3) + 2 = (n+3)\lg n + 2$ for $k > 0$.

Proof We have:

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = \sum_{i=1}^{n+1} \lfloor \lg i \rfloor - \lfloor \lg(n+1) \rfloor =$$

[by (1)]

$$= (n+2)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2 - \lfloor \lg(n+1) \rfloor = (n+1)\lfloor \lg(n+1) \rfloor - 2^{\lfloor \lg(n+1) \rfloor + 1} + 2.$$

□

It turns out that the value of

$$x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1}$$

(which is a part of the right-hand side of (2) for $x = n+1$) oscillates between

$$x(\lg x - 2)$$

and

$$x(\lg x - 2 + 0.08607133205593432).$$

If $x = 2^k$ then $\lg x = k$, which is an integer number, so $\lfloor \lg x \rfloor = \lg x$ and, indeed,

$$x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1} = x \lg x - 2 \times 2^{\lg x} = x \lg x - 2x = x(\lg x - 2).$$

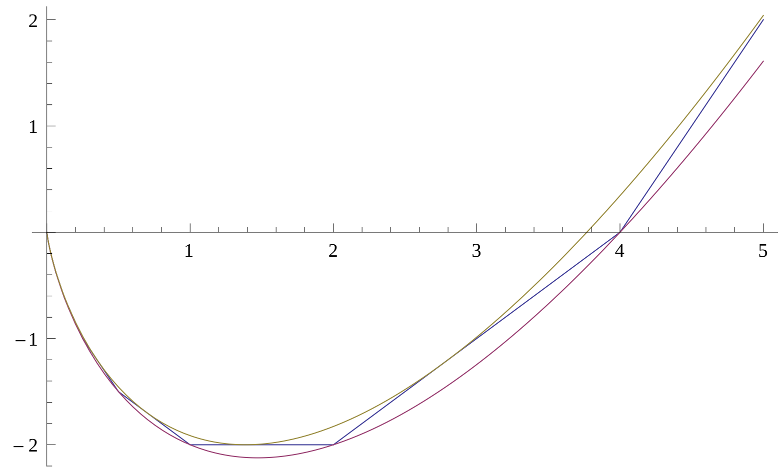

Figure 3: Functions $x(\lg x - 2)$ (bottom), $x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1}$ (middle), and $x(\lg x - 2 + 0.08607133205593432)$ (top).

We will show that for $x \neq 2^k$, value of $x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1}$ is (only slightly) larger than $x(\lg x - 2)$ and not larger than $x(\lg x - 2 + 0.08607133205593432)$, as the Figure 3 shows.

Theorem 1.3 *For every $x > 0$,*

$$x\lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1} = x(\lg x + \alpha(x) - 2), \tag{3}$$

where α is given by:

$$\alpha(x) = 2 - \varphi - 2^{1-\varphi} \text{ and } \varphi = \lg\, x - \lfloor \lg\, x \rfloor.$$

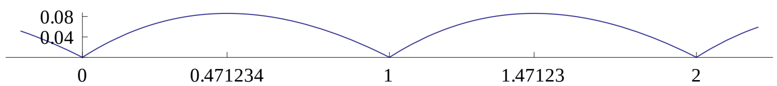

Figure 4: Graph of $\alpha(x) = 2-(y-\lfloor y\rfloor)-2^{1-(y-\lfloor y\rfloor)}$ as a function of $y = \lg x$.

Proof. Substituting definition of φ to the definition of α, we obtain:

$$\alpha(x) = 2 - (\lg\, x - \lfloor \lg\, x \rfloor) - 2^{1-(\lg\, x - \lfloor \lg\, x \rfloor)},$$

or

$$\alpha(x) = 2 - \lg\, x + \lfloor \lg\, x \rfloor - \frac{2^{\lfloor \lg\, x \rfloor + 1}}{2^{\lg\, x}},$$

or

$$\lg\, x + \alpha(x) - 2 = \lfloor \lg\, x \rfloor - \frac{2^{\lfloor \lg\, x \rfloor + 1}}{x},$$

or

$$x(\lg\, x + \alpha(x) - 2) = x\lfloor \lg\, x \rfloor - 2^{\lfloor \lg\, x \rfloor + 1}.$$

□

Theorem 1.4 *For every $x > 0$,*

$$x\lceil \lg x\rceil - 2^{\lceil \lg x\rceil} = x(\lg x + \varepsilon(x) - 1), \tag{4}$$

where ε is given by:

$$\varepsilon(x) = 1 + \theta - 2^{\theta} \text{ and } \theta = \lceil \lg\ x\rceil - \lg\ x.$$

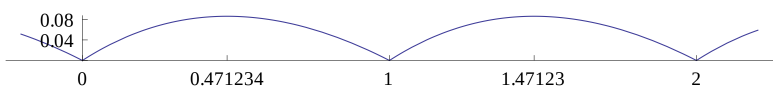

Figure 5: Graph of $\varepsilon(x) = 1 + (\lceil y\rceil - y) - 2^{\lceil y\rceil - y}$ as a function of $y = \lg x$.

Proof. Substituting definition of θ to the definition of ε, we obtain:

$$\varepsilon(x) = 1 + \lceil \lg\ x\rceil - \lg\ x - 2^{\lceil \lg\ x\rceil - \lg\ x},$$

or

$$\lg\ x + \varepsilon(x) - 1 = \lceil \lg\ x\rceil - \frac{2^{\lceil \lg\ x\rceil}}{2^{\lg\ x}},$$

or

$$\lg\ x + \varepsilon(x) - 1 = \lceil \lg\ x\rceil - \frac{2^{\lceil \lg\ x\rceil}}{x},$$

or

$$x(\lg\ x + \varepsilon(x) - 1) = x\lceil \lg\ x\rceil - 2^{\lceil \lg\ x\rceil}.$$

□

Theorem 1.5 *For every x,*

$$\varepsilon(x) = \alpha(x), \tag{5}$$

where ε is given by:

$$\varepsilon(n) = 1 + \theta - 2^{\theta} \textit{ and } \theta = \lceil \lg x \rceil - \lg x,$$

and α is given by:

$$\alpha(x) = 2 - \varphi - 2^{1-\varphi} \textit{ and } \varphi = \lg x - \lfloor \lg x \rfloor.$$

Proof. If for some integer k, $x = 2^k$ then

$$\lg x = k = \lceil \lg x \rceil = \lfloor \lg x \rfloor.$$

In such a case,

$$\theta = \varphi = 0,$$

so

$$\varepsilon(x) = 1 + \theta - 2^{\theta} = 1 + 0 - 2^0 = 0 = 2 - 0 - 2^{1-0} = 2 - \varphi - 2^{1-\varphi} = \alpha(n).$$

Thus $\varepsilon(n) = \alpha(x)$ in such a case.

Otherwise, $\lg x$ is not an integer, so

$$\lceil \lg x \rceil = \lfloor \lg x \rfloor + 1,$$

and

$$\theta = \lceil \lg\ x \rceil - \lg\ x = \lfloor \lg\ x \rfloor + 1 - \lg\ x = 1 - (\lfloor \lg x \rfloor - \lg\ x) = 1 - \varphi.$$

From this we conclude

$$\varepsilon(x) = 1 + \theta - 2^{\theta} = 1 + 1 - \varphi - 2^{1-\varphi} = 2 - \varphi - 2^{1-\varphi} = \alpha(x).$$

□

Corollary 1.6 *For every $x > 0$,*

$$x \lfloor \lg x \rfloor - 2^{\lfloor \lg x \rfloor + 1} = x(\lg x + \varepsilon(x) - 2), \tag{6}$$

where ε is given by:

$$\varepsilon(x) = 1 + \theta - 2^{\theta} \text{ and } \theta = \lceil \lg\ x \rceil - \lg\ x.$$

Proof by direct application of Theorem 1.5 to Theorem 1.3. □

Corollary 1.7 *For every natural number $n \geq 1$,*

$$(n+1) \lg \frac{n+1}{4} + 2 \leq \sum_{i=1}^{n} \lfloor \lg i \rfloor \leq (n+1)(\lg \frac{n+1}{4} + 0.08607133205593432) + 2. \tag{7}$$

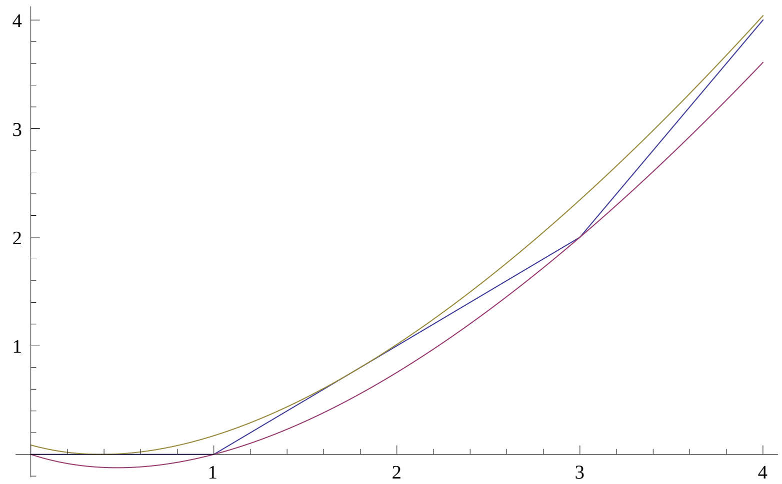

Figure 6: Functions $(n+1)\lg\frac{n+1}{4}+2$ (bottom), $(n+1)\lfloor\lg(n+1)\rfloor - 2^{\lfloor\lg(n+1)\rfloor+1}+2$ (middle, same as $\sum_{i=1}^{n}\lfloor\lg i\rfloor$ for all integer $n\geq 1$), and $(n+1)(\lg\frac{n+1}{4}+0.08607133205593432)+2$ (top).

Proof. Putting $x=n+1$ in equality (6) of Theorem 1.3 yields:

$$(n+1)\lfloor\lg(n+1)\rfloor - 2^{\lfloor\lg(n+1)\rfloor+1}+2 = (n+1)(\lg(n+1)+\alpha(\lg(n+1))-2)+2 =$$

$$= (n+1)(\lg(n+1)+\alpha(\lg(n+1))-\lg 4)+2 = (n+1)(\lg\frac{n+1}{4}+\alpha(\lg(n+1)))+2.$$

Hence, by the equality (2) of Theorem 1.2,

$$\sum_{i=1}^{n} \lfloor \lg i \rfloor = (n+1)(\lg \frac{n+1}{4} + \alpha(\lg(n+1))) + 2. \tag{8}$$

Since, as Figure 4 shows, $0 \leq \alpha(x) \leq 0.08607133205593432$ (use Mathematica to find the minimum and the maximum of $\alpha(x)$ or refer to the Lemma 1.8), we conclude (7). □

Lemma 1.8 *For every* y*,*

$$0 \leq \alpha(y) \leq 0.08607133205593432, \tag{9}$$

where $\alpha(y) = 2 - (y - \lfloor y \rfloor + \frac{2}{2^{y-\lfloor y \rfloor}})$ *(see Figure 4).*

Proof. Function $\alpha(y)$ is periodic with period 1, that is for every y,

$$\alpha(y) = \alpha(y+1).$$

Therefore, in order to find the minimum and the maximum of $\alpha(x)$ it suffices to find it on the closed interval $[0, 1]$.

$$\alpha(0) = \alpha(1) = 0.$$

For any $0 \leq y \leq 1$, $\lfloor y \rfloor = 0$, so

$$\alpha(y) = 2 - (y - \lfloor y \rfloor + \frac{2}{2^{y-\lfloor y \rfloor}}) = 2 - (y + \frac{2}{2^y}) = 2 - y - 2^{1-y}.$$

Let us compute the derivative $\alpha'(y)$ of $\alpha(y)$ on the open interval $(0, 1)$.

$$\alpha'(y) = [2 - y - 2^{1-y}]' = 2' - y' - [2^{1-y}]' =$$
$$= 0 - 1 - \ln 2 \times 2^{1-y} \times [1 - y]' = -1 - \ln 2 \times 2^{1-y} \times (-1) = 2^{1-y} \ln 2 - 1.$$

Let us solve the equation

$$\alpha'(y) = 0$$

for $0 < y < 1$. We have:

$$\alpha'(y) = 2^{1-y} \ln 2 - 1 = 0,$$

so

$$2^{1-y} \ln 2 = 1,$$

or

$$2^{1-y} = \frac{1}{\ln 2},$$

or

$$2^{1-y} = \lg e,$$

or

$$\lg 2^{1-y} = \lg \lg e,$$
$$1 - y = \lg \lg e,$$

or

$$y = 1 - \lg \lg e \approx 0.4712336270551024.$$

Substituting $y = 1 - \lg\lg e$ in $\alpha(y) = 2 - y - 2^{1-y}$, we obtain

$$\alpha(1-\lg\lg e) = 2-(1-\lg\lg e)-2^{1-(1-\lg\lg e)} = 1+\lg\lg e-2^{\lg\lg e} = 1+\lg\lg e-\lg e =$$

$$= 1 - \lg e + \lg\lg e \approx 0.08607133205593431,$$

as $\lg e \approx 1.4426950408889634$ and $\lg\lg e \approx 0.5287663729448976$.

From the above calculations, we conclude that 0 is the minimum and 0.08607133205593431 is the approximate maximum of function $\alpha(y)$ on the closed interval $[0, 1]$.

Hence, for all y.

$$0 \leq \alpha(y) \leq 0.08607133205593432.$$

□

Note. The constant

$$1 - \lg e + \lg\lg e \approx 0.08607133205593431$$

has been known as the *Erdös constant* δ. Erdös used it around 1955 in order to establish an asymptotic upper bound for the number $M(k)$ of different numbers in a multiplication table of size $k \times k$ by means of the following limit:

$$\lim_{k\to\infty} \frac{\ln \frac{k\times k}{M(k)}}{\ln\ln(k \times k)} = \delta.$$

In other words,

$$M(k) \sim \frac{k^2}{(2\ln k)^{0.08607133205593431}}.$$

Corollary 1.9 *For every natural number $n \geq 1$,*

$$(n+1)\lg\frac{n+1}{4}+2 \leq \sum_{i=1}^{n}\lfloor \lg i \rfloor \leq (n+1)(\lg\frac{n+1}{4}+0.08607133205593432)+2. \tag{10}$$

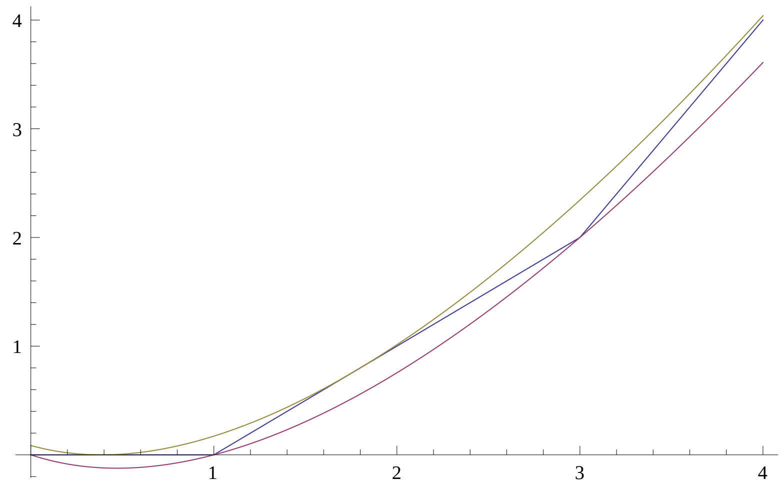

Figure 7: Functions $(n+1)\lg\frac{n+1}{4}+2$, $(n+1)\lfloor\lg(n+1)\rfloor-2^{\lfloor\lg(n+1)\rfloor+1}+2$, and $(n+1)(\lg\frac{n+1}{4}+0.08607133205593432)+2$.

2 A sum of ceilings of consecutive logarithms

Theorem 2.1 *For every natural number $n \geq 1$,*

$$\sum_{i=1}^{n}\lceil\lg i\rceil = n\lceil\lg n\rceil - 2^{\lceil\lg n\rceil}+1. \tag{11}$$

Note. The right-hand side of (11) is a continuous function.

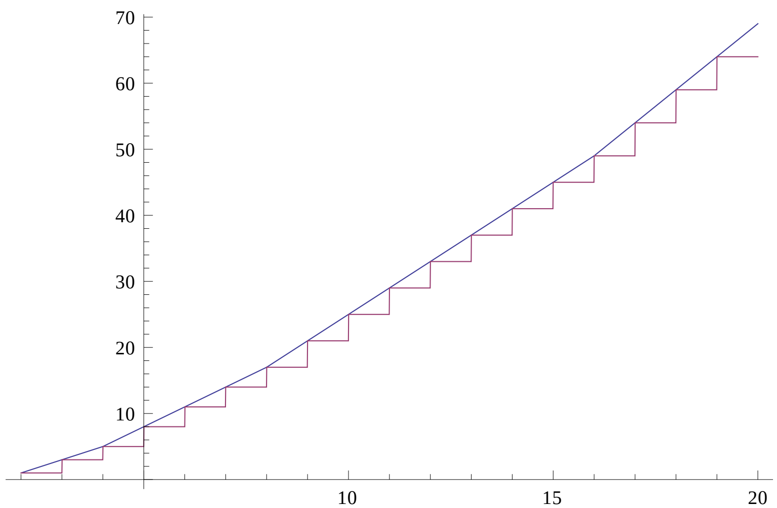

Figure 8: Functions $\sum_{i=1}^{n}\lceil\lg i\rceil$ and $n\lceil\lg n\rceil - 2^{\lceil\lg n\rceil} + 1$.

Note. Function $F(n) = \sum_{i=1}^{n}\lceil\lg i\rceil = n\lceil\lg n\rceil - 2^{\lceil\lg n\rceil} + 1$ is a linear interpolation of itself restricted to $n = 2^k$. In particular, it is a linear interpolation of the function $G(k) = (k-1)2^k + 1 = n(\lg n - 1) + 1$.

Proof We have

$$\sum_{i=1}^{n}\lceil\lg i\rceil = \sum_{i=2}^{n}\lceil\lg i\rceil =$$

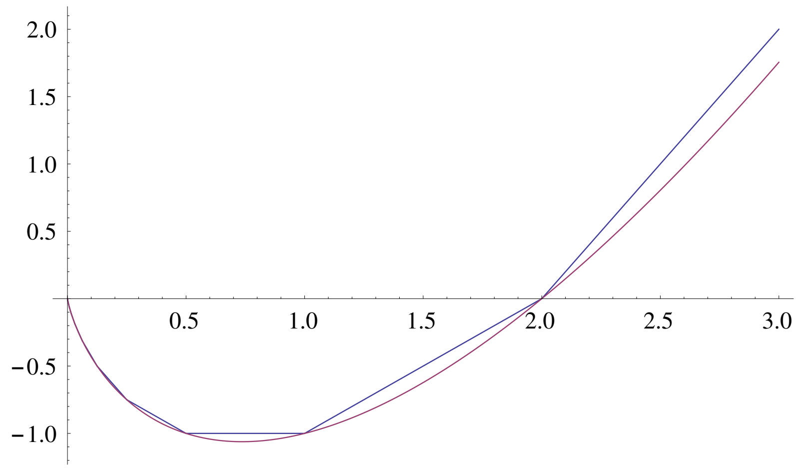

Figure 9: Functions $n\lceil\lg n\rceil - 2^{\lceil\lg n\rceil}$ (top) and $n(\lg n - 1)$ (bottom); the former is a linear interpolation of the latter between points $n = 2^{\lceil\lg n\rceil}$.

[by $\lceil\lg i\rceil = \lfloor\lg(i-1)\rfloor + 1$]

$$= \sum_{i=2}^{n}(\lfloor\lg(i-1)\rfloor+1) = \sum_{i=2}^{n}\lfloor\lg(i-1)\rfloor+\sum_{i=2}^{n}1 = \sum_{i=2}^{n}\lfloor\lg(i-1)\rfloor+n-1 = \sum_{i=1}^{n-1}\lfloor\lg(i)\rfloor+n-1 =$$

[by (1)]

$$= n\lfloor\lg(n-1)\rfloor - 2^{\lfloor\lg(n-1)\rfloor+1} + 2 + n - 1 =$$

[by $\lfloor \lg(n-1) \rfloor = \lceil \lg n \rceil - 1$]

$= n(\lceil \lg n \rceil - 1) - 2^{\lceil \lg n \rceil - 1 + 1} + 2 + n - 1 = n\lceil \lg n \rceil - n - 2^{\lceil \lg n \rceil} + 2 + n - 1 = n\lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1.$

(See also [Knu97].) □

Corollary 2.2 *For every natural number $n \geq 1$,*

$$n \lg \frac{n}{2} + 1 \leq \sum_{i=1}^{n} \lceil \lg i \rceil \leq n(\lg \frac{n}{2} + 0.08607133205593432) + 1. \tag{12}$$

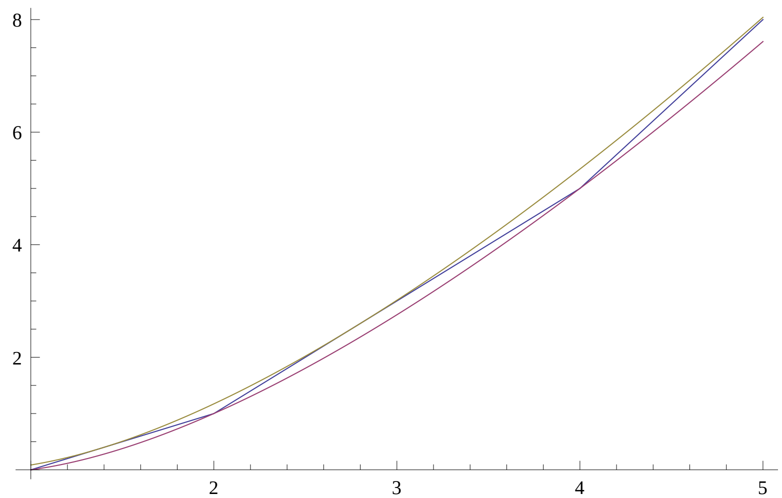

Figure 10: Functions $n \lg \frac{n}{2} + 1$, $n\lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1$, and $n(\lg \frac{n}{2} + 0.08607133205593432) + 1$.

Theorem 2.3 *For every natural number $n \geq 1$,*

$$\sum_{i=1}^{n} \lceil \lg i \rceil = n \lceil \lg(n+1) \rceil - 2^{\lceil \lg(n+1) \rceil} + 1. \tag{13}$$

Note. The right-hand side of (13) is not a continuous function.

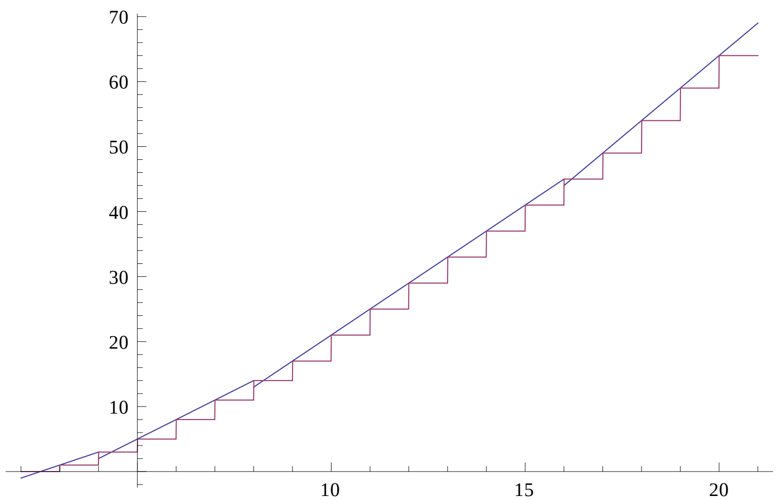

Figure 11: Functions $\sum_{i=1}^{n} \lceil \lg i \rceil$ and $n \lceil \lg(n+1) \rceil - 2^{\lceil \lg(n+1) \rceil} + 1$.

Proof We have:

$$\sum_{i=1}^{n} \lceil \lg i \rceil = \sum_{i=1}^{n+1} \lceil \lg i \rceil - \lceil \lg(n+1) \rceil =$$

[by (11)]

$$= (n+1)\lceil \lg(n+1)\rceil - 2^{\lceil \lg(n+1)\rceil} + 1 - \lceil \lg(n+1)\rceil = n\lceil \lg(n+1)\rceil - 2^{\lceil \lg(n+1)\rceil} + 1.$$

□

Theorem 2.4 *For every natural number $n \geq 1$,*

$$\sum_{i=1}^{n} \lceil \lg i \rceil = n\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + n + 1. \tag{14}$$

Proof By (13) we have:

$$\sum_{i=1}^{n} \lceil \lg i \rceil = \sum_{i=1}^{n+1} \lceil \lg i \rceil - \lceil \lg(n+1)\rceil = n\lceil \lg(n+1)\rceil - 2^{\lceil \lg(n+1)\rceil} + 1 =$$

$$= n(\lfloor \lg n \rfloor + 1) - 2^{\lfloor \lg n \rfloor + 1} + 1 = n\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + n + 1.$$

□

Theorem 2.5 *For every natural number $n \geq 1$,*

$$\sum_{i=1}^{n} \lceil \lg i \rceil = n(\lg n + \varepsilon(n)) - n + 1. \tag{15}$$

Proof By (14) we have:

$$\sum_{i=1}^{n} \lceil \lg i \rceil = n\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + n + 1 =$$

[by (6)]

$$= n(\lg n + \varepsilon(n) - 2) + n + 1 = n(\lg n + \varepsilon(n)) - 2n + n + 1 = n(\lg n + \varepsilon(n)) - n + 1.$$

□

3 A sum of the differences

Theorem 3.1 *For every natural number $n \geq 1$,*

$$\sum_{i=1}^{n} (\lceil \lg i \rceil - \lfloor \lg i \rfloor) = n - \lceil \lg(n+1) \rceil = n - \lfloor \lg n \rfloor - 1. \tag{16}$$

Proof. We have

$$\sum_{i=1}^{n} (\lceil \lg i \rceil - \lfloor \lg i \rfloor) = \sum_{i=1}^{n} \lceil \lg i \rceil - \sum_{i=1}^{n} \lfloor \lg i \rfloor =$$

[by (1) and (14)]

$$n\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + n + 1 - ((n+1)\lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor + 1} + 2) = n - \lfloor \lg n \rfloor - 1.$$

□

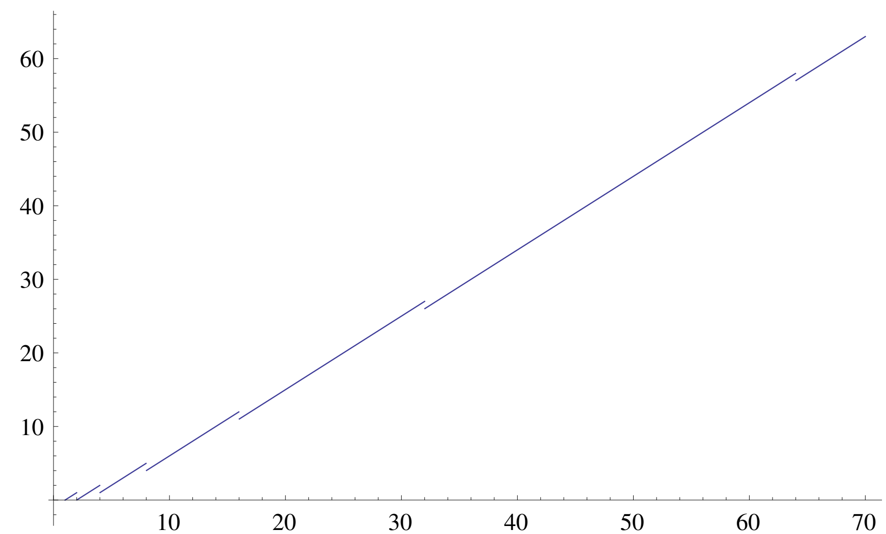

Figure 12: Function $n - \lfloor \lg n \rfloor - 1$.

References

[Knu97] Donald E. Knuth. *The Art of Computer Programming*, volume 3. Addison-Wesley Publishing, 2nd edition, 1997.