Sums of floors and ceilings of consecutive fractions

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The formulas in this paper are used in derivation of a more exact form of the Master Theorem.

1 A sum of floors of consecutive fractions

Theorem 1.1 *For every natural number n and every positive natural number m,*

$$\sum_{i=0}^{m-1} \lfloor \frac{n+i}{m} \rfloor = n.$$

Proof. Let $n = km + l$, where $0 \leq l < m$.

We have

$$\lfloor \frac{n+i}{m} \rfloor = \lfloor \frac{km+l+i}{m} \rfloor = \lfloor k + \frac{l+i}{m} \rfloor = k + \lfloor \frac{l+i}{m} \rfloor.$$

Therefore,

$$\sum_{i=0}^{m-1} \lfloor \frac{n+i}{m} \rfloor = mk + \sum_{i=0}^{m-1} \lfloor \frac{l+i}{m} \rfloor = mk + \sum_{i=m-l}^{m-1} \lfloor \frac{l+i}{m} \rfloor = mk + \sum_{i=m-l}^{m-1} 1 = mk + l = n.$$

2 A sum of ceilings of consecutive fractions

Theorem 2.1 *For every natural number* n *and every positive natural number* m*,*

$$\sum_{i=0}^{m-1} \lceil \frac{n-i}{m} \rceil = n.$$

Proof.Let $n = km + l$, where $0 \leq l < m$.

We have

$$\lceil \frac{n-i}{m} \rceil = \lceil \frac{km+l-i}{m} \rceil = \lceil k + \frac{l-i}{m} \rceil = k + \lceil \frac{l-i}{m} \rceil.$$

Therefore,

$$\sum_{i=0}^{m-1} \lceil \frac{n-i}{m} \rceil = mk + \sum_{i=0}^{m-1} \lceil \frac{l-i}{m} \rceil = mk + \sum_{i=0}^{l-1} \lceil \frac{l-i}{m} \rceil = mk + \sum_{i=0}^{l-1} 1 = mk + l = n.$$