TREE CHARACTERIZATION THM

Let T be a graph with n vertices. Any two of the following three properties of T imply the third property of T.

(i) T is connected
(ii) T is acyclic
(iii) T has $n-1$ edges

PROOF.

(i) & (ii) $\Rightarrow$ (iii)

By induction on n.

<u>Basis step</u> $n=1$

An acyclic graph with 1 vertex must have no edges. So, the number of edges is $0 = 1-1 = n-1$.

<u>Inductive step</u>

Assume that (i)&(ii) ⇒ (iii) holds for any graph T with n vertices. ($n \geq 1$) We will prove that it also holds for any graph U with $n+1$ vertices.

Let U be a connected acyclic graph with $n+1$ vertices. Since U is a finite tree, it must have a leaf w.
w must have only one edge e incident on it.

Let's remove w and e from U.

The resulting graph S remains acyclic (removal of an edge cannot add a cycle) and connected (w was the only part of U that was connected to the rest of it via edge e). S has n vertices, so, by the inductive hypothesis, it has n-1 edges. Therefore, U has $(n-1)+1 = n$ edges. (U has n+1 vertices).

Therefore, U satisfies (i) & (ii) $\Rightarrow$ (iii).

This completes the inductive step.

(i) & (iii) $\Rightarrow$ (ii)

Let T be connected, have $n-1$ edges and have a cycle.

Let U be a result of removal one edge $e=(v,w)$ from a cycle in T. Because e was a part of cycle, the rest of the cycle is a path in U from v to w. So after removal of e, U is still connected, has n vertices, and $n-2$ edges. This contradicts the already proved (i) & (ii) $\Rightarrow$ (iii).

(ii) & (iii) $\Rightarrow$ (i).

Let T be acyclic, have $n-1$ edges, and be not connected.

Adding an edge (v,w) to T such that v and w belong to different connected components of T produces an acyclic graph T_1 with n edges. If T_1 is still not connected, one can add another edge that connects vertices of different connected components of T_1 and the resulting graph T_2 has $n+1$ edges and is acyclic.

The above process cannot be continued indefinitely (T was finite), so it must eventually produce an acyclic and connected graph with n or more edges. This contradicts the already proved (i) & (ii) ⇒ (iii).

This completes the proof. □