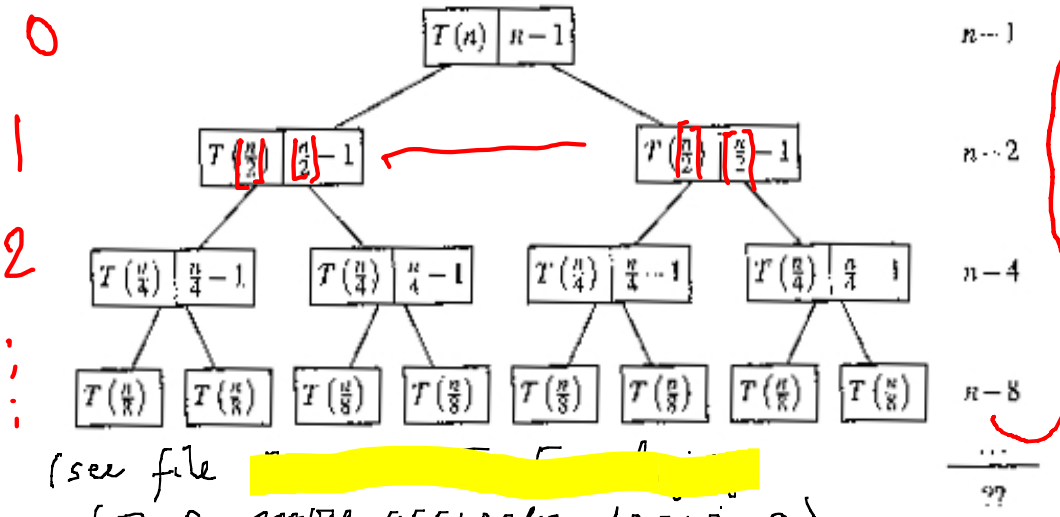
$W(n)$: the worst-case number of comps done by MergeSort

Note Title 3/6/2014

$W(n)$: the worst-case number of comps done by MergeSort

Note Title 2/21/2012



(see file [redacted] for a more accurate drawing)

Figure 4.14 Recursion tree for Mergesort. Wherever a node size parameter is odd, the left child size is rounded up and the right child size is rounded down.

at level i:
$n - 2^i$ comps
⋮

down to the last level k where any merging takes place:
$n - 2^k$ comps $(2^k \le n)$

What is the largest k here?

$$\max\{k \mid 2^k \le n\} = \max\{k \mid \lg 2^k \le \lg n\} =$$

$$= \max\{k \mid k \le \lg n\} = \lfloor \lg n \rfloor$$

Recall: $\max\{n \mid n \le x\} = \lfloor x \rfloor$

So, the total number of comps in all merges is (exactly!):

$$W(n) = \sum_{i=0}^{\lfloor \lg n \rfloor} \left(n - 2^i\right) = \sum_{i=0}^{\lfloor \lg n \rfloor} n - \sum_{i=0}^{\lfloor \lg n \rfloor} 2^i =$$

$$= \left(\lfloor \lg n \rfloor + 1\right) n - \left(2^{\lfloor \lg n \rfloor + 1} - 1\right) =$$

$$= n\left(\lfloor \lg n \rfloor + 1\right) - 2^{\lfloor \lg n \rfloor + 1} + 1 =$$

(using $\lfloor \lg n \rfloor + 1 = \lceil \lg (n+1) \rceil$)

$$= W(n) = \boxed{n \lceil \lg (n+1) \rceil - 2^{\lceil \lg (n+1) \rceil} + 1.}$$

The number of comps by MergeSort in the worst case while sorting n elements

$$= n \lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1 = \sum_{i=1}^{n} \lceil \lg i \rceil$$

A neat formula

(the last 2 formulas are [redacted])

The above formula is the same as in the textbook (p 177, the last line of equation (4.6)), except that ours has been derived exactly w/o any approximations or simplifying assumptions.

Here is a formula without the floors and ceilings that uses function $\mathcal{E}$ defined before:

$$W(n) = n(\lg n + \mathcal{E}(n) - 1) + 1$$

$\alpha \cdot \lg \alpha$ in the textbook

Since $\mathcal{E}(n)$ ranges between 0 and 0.08607..., the above formula yields the correct bounds:

$$n \lg n - n + 1 \leq W(n) < n \lg n - 0.913\, n + 1$$

(the constant 0.914 used in the textbook was an error.)