Preservation Properties in Deductive Data Bases

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Abstract

This paper constitutes a brief study of the following scheme DA_Γ of default reasoning:

> Given set of premises Σ, assume φ if and only if, for every $\gamma \in \Gamma$, $\Sigma \cup \{\varphi\}$ does not entail γ unless Σ does,

where Γ and Σ are sets of clauses, and φ is a clause. The problem of $\forall$-completeness of DA_Γ with respect to finest model semantics is investigated. A method of deriving $\forall$-completeness of $DA_{\Gamma_1 \cap \Gamma_2}$ assuming DA_{Γ_1} and DA_{Γ_2} are known to be $\forall$-complete is presented. This method yields $\forall$-completeness of instances of DA_Γ with respect to totally minimal semantics, discriminant minimal semantics, Herbrand semantics, and minimal Herbrand semantics. It is shown that the last of these completeness results is strictly stronger than analogous characterization using an extension of the generalized closed-world assumption *GCWA*.

1 Introduction

In this paper, we investigate instances of the following scheme D of default reasoning:

> If the assumption φ doesn't introduce unwanted consequences to the set of premises Σ then assume φ.

Although scheme D does not seem to cover all the default rules discussed in professional literature, it has a number of intriguing applications in deductive data bases and in logic programming. For example, all published versions of the *closed-world assumption* are, in fact, instances of this scheme. If the set of "unwanted consequences", say Γ, is finite then D may be expressed in the form of a set of Reiter style *default rules of inference*:

$$\frac{\Gamma_0 \cup \{M(\varphi \wedge \neg\gamma_1), ..., M(\varphi \wedge \neg\gamma_k)\}}{\varphi} \tag{1}$$

where Γ_0 ranges over consistent subsets of Γ, and $\gamma_1, ..., \gamma_k$ are all elements of $\Gamma \setminus \Gamma_0$. For instance, if formula *false* is the only unwanted consequence then (1) may be reduced to a single *normal default*:

$$\frac{\{M(\varphi)\}}{\varphi}.$$

To express D in the general case of infinite Γ, one must admit infinitary default rules of inference. For this reason, we will utilize a different way of formalizing default schemes of this kind.

Using the default scheme D in the context of reasoning from an arbitrary set of premises of Σ involves a subtlety. If Σ itself entails consequences that are unwanted, then the requirement "φ doesn't introduce unwanted consequences" should be understood as "φ doesn't entail any unwanted consequence, unless this unwanted consequence is already entailed in Σ". This observation, given the set Γ of all unwanted consequences, leads to the formalization of D by the following postulate:

> Assume φ if and only if, for every $\gamma \in \Gamma$, $\Sigma \cup \{\varphi\}$ does not entail γ unless Σ does.

We will refer to it as the Γ-*default assumption*, and denote the set of all φ that satisfy the above definition by $DA_\Gamma(\Sigma)$.

The consistency of $DA_\Gamma(\Sigma)$ seems to be a serious problem that is reportedly common to all kinds of default rules. In particular, it may happen that each of a pair φ, ψ of inconsistent sentences separately does not add any unwanted consequences to Σ, and then both of them are included in $DA_\Gamma(\Sigma)$, making $DA_\Gamma(\Sigma)$ inconsistent. In this paper, we have managed to avoid this problem by restricting both Σ and $DA_\Gamma(\Sigma)$ to sentences which do not reveal such pathological behavior. Most notably, clauses, or universally quantified sentences, do not cause this type of a problem for appropriately chosen Γ. As a result of this precaution, the instances of the Γ-default assumption which are studied here may be treated as rules of inference and used whenever applicable, without contradicting the other uses of the same instance of Γ-default assumption. This desirable proof-theoretic property frees us from fixed-point equations which seem inevitable for the default rules that do not possess this feature.

Consistency of $DA_\Gamma(\Sigma)$ creates the possibility of characterizing it in terms of its models, which is an object of our primary concern. A question which we will attempt to address to the fullest extent possible is:

> Given Γ, what semantics assigns to every Σ from the domain of DA_Γ the class of first-order models of $DA_\Gamma(\Sigma)$?

We will seek candidates for such semantics among models of Σ which are *finest* with respect to certain preference relations (called refinements in this paper) that are characteristic for certain instances of Γ. This approach leads to a number of completeness results of the form:

> $\varphi \in DA_\Gamma(\Sigma)$ if and only if every finest model of Σ is a model of φ.

The instances of Γ we investigate in this paper are the sets of clauses positive beyond equality, clauses positive in equality, ground clauses, the intersections of these sets, and the set of clauses negative in equality. The corresponding minimal semantics are: predicate-minimal semantics, discriminant semantics, domain-minimal semantics, and their combinations. For example, if Γ consists of all positive ground clauses then DA_Γ is $\forall$-complete with respect to minimal Herbrand (i.e., predicate-minimal, domain-minimal, and discriminant) semantics; a result whose extreme instances (e.g., no $=$ in the language) were known for some time in a weaker form. It is our hope that our uniform analysis of these cases will clarify the actual relationships between different patterns of model minimization described in the literature.

2 Prerequisites from Logic and Algebra

We apply the standard terminology and notation of first-order model theory, a good account of which can be found in [1], Chap. A2. We restrict ourselves to a first-order language L with logical connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*not*), $\forall$ (*for all*) and $\exists$ (*there exists*), and treat all other connectives as abbreviations. In addition to predicates, L contains constants and the equality symbol $=$[1], and may contain function symbols. We identify L with the set of its all formulas. To avoid confusion with connectives of L, we apply beyond L the informal quantifiers $\bigwedge$ and $\bigvee$ as abbreviations for *for all* and *there exists*. For instance,

$$\bigwedge_{\varphi \in L} \forall x \varphi \in L$$

means:

> For every formula φ of L, $\forall x \varphi$ is a formula of L.

[1] The case of L without $=$ is addressed in Section 6.

A formula φ is *atomic* if and only if no logical connective appears in φ. For example, $x = f(d)$ is an atomic formula. A $\forall$-formula is a formula of the form $\forall x_1 \forall x_2 ... \forall x_n \varphi$ (abbreviated as $\forall \vec{x} \varphi$), where φ is quantifier-free. Similarly, an $\exists$-formula is a formula of the form $\exists x_1 \exists x_2 ... \exists x_n \varphi$ (abbreviated as $\exists \vec{x} \varphi$), where φ is quantifier-free. A *sentence* is a formula without free (i.e., unquantified) variables. *Atom* denotes the set of all atomic sentences of L; QF denotes the set of all quantifier-free sentences of L; and $\forall$ and $\exists$, when used in appropriate context, denote the set of all $\forall$-sentences and the set of all $\exists$-sentences of L, respectively.

$\forall$-sentences seem to play a very important role in artificial intelligence. In particular, every clause is logically equivalent to a $\forall$-sentence. Also, every $\forall$-sentence is logically equivalent to a finite set of clauses. Therefore, in this paper $\forall$-sentences are given special attention. Moreover, we identify sets of clauses with subsets of $\forall$, and sets of ground clauses with subsets of QF. Obviously, this can be done without loss of generality.

$\mathbf{S}_L$ denotes the class of all first-order structures for L. We usually denote its elements by $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, ...$, and their respective domains by $A, B, M, N, ...,$. If F is a function symbol of L then $F^{\mathcal{M}}$ denotes the corresponding function in $\mathcal{M}$. Analogously, $R^{\mathcal{M}}$ denotes the relation in $\mathcal{M}$ corresponding to the predicate R. The *satisfaction relation* is denoted by $\models$, so that $\mathcal{M} \models \varphi[s]$ means that formula φ is satisfied in structure $\mathcal{M}$ by assignment s on its free variables (formally, s is a function from variables of L into the domain M of $\mathcal{M}$). If φ is satisfied in $\mathcal{M}$ by every assignment s then $\mathcal{M} \models \varphi$ is used instead of $\mathcal{M} \models \varphi[s]$. In such a case we say that φ is true in $\mathcal{M}$. If φ is a sentence of L then for every assignment s in M, $\mathcal{M} \models \varphi[s]$ is synonymous with $\mathcal{M} \models \varphi$. We use $\mathcal{M} \subseteq \mathcal{N}$, to mean that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$, if and only if $M \subseteq N$, and for every predicate or function symbol F of L, $F^{\mathcal{M}} = F^{\mathcal{N}} \restriction M$, where "$\restriction$" means "restricted to".

A *theory* Σ in L is a set of sentences of L. Here we usually denote formulas by lower-case Greek letters, and theories by upper-case Greek letters. We write $\mathcal{M} \models \Sigma$ to express that for every $\varphi \in \Sigma$, $\mathcal{M} \models \varphi$. $\mathcal{M} \models \Sigma$ means that all elements of Σ are true in $\mathcal{M}$, that is, the structure $\mathcal{M}$ is a model of Σ. We denote the class of all models of Σ by $\mathbf{Mod}(\Sigma)$. We make an implicit use of the completeness theorem of first-order logic by applying $\Sigma \vdash \varphi$ in the sense $\mathbf{Mod}(\Sigma) \subseteq \mathbf{Mod}(\varphi)$, so that $\Sigma \vdash \varphi$ means that φ is true in every model of Σ. If f is a function from subclasses of $\mathbf{S}_L$ into subclasses of $\mathbf{S}_L$ then $\vdash_f$ is a relation defined by:

$$\bigwedge_{\Sigma \subseteq L} \bigwedge_{\varphi \in L} \Sigma \vdash_f \varphi \text{ if and only if } \mathbf{Mod}(\Sigma) \cap f(\mathbf{Mod}(\Sigma)) \subseteq \mathbf{Mod}(\varphi). \qquad (2)$$

$\Sigma \vdash_f \varphi$ means that φ is true in all models of Σ that are in $f(\mathbf{Mod}(\Sigma))$, but not necessarily in all models of Σ. Relations $\vdash$ and $\vdash_f$ are called *entailments*.

We use $Cn(\Sigma)$ to denote the set $\{\varphi \mid \Sigma \vdash \varphi\}$ of all first-order consequences of Σ within the language L. We say that Σ_1 is *logically equivalent* to Σ_2 if and

only if $Cn(\Sigma_1) = Cn(\Sigma_2)$, or equivalently, if and only if $\mathbf{Mod}(\Sigma_1) = \mathbf{Mod}(\Sigma_2)$. Moreover if Γ is a set of sentences of L then we use Σ_Γ to denote the set $Cn(\Sigma) \cap \Gamma$. In particular, Σ_L coincides with $Cn(\Sigma)$.

The notion of *relative positiveness* and *negativeness* are the key concepts which allow for proper articulation of default assumptions that are $\forall$-complete with respect to various versions of minimal semantics.

Let $\Re$ be a set of predicates of L. Our intention is that $\Re$ contains all predicates which are being minimized, while the predicates not in $\Re$ remain constant. A formula is *positive* in a symbol R if and only if it does not contain an appearance of R in a scope of a negation[2]. It is *positive* if and only if it is positive in every predicate in $\Re$ and in the equality symbol $=$. A formula is *negative* in R if and only if it is a negation of a formula which is positive in R. In this paper, $Pos_{\{=\}'}$ denotes the set of all $\forall$-sentences which are positive in all predicates in $\Re$ (symbol "$'$" in $Pos_{\{=\}'}$ indicates that $=$ is excluded from the requirement of positiveness), $Pos_{\{=\}}$ denotes the set of all $\forall$-sentences which are positive in $=$, Pos (the intersection of $Pos_{\{=\}'}$ and $Pos_{\{=\}}$) denotes the set of all positive $\forall$-sentences, and $Neg_{\{=\}}$ denotes the set of all $\forall$-sentences which are negative in $=$. Let us note that elements of Pos may contain negative occurrences of predicates from beyond $\Re$.

A *conservative homomorphism* on $\mathcal{M}$ into $\mathcal{N}$ is a mapping $f : M \rightarrow N$ such that for each atomic formula φ without an occurrence of the symbol $=$, and assignment s in M,

$$\mathcal{M} \models \varphi[s] \textbf{ iff } \mathcal{N} \models \varphi[s \circ f],$$

and for each term t and assignment s in M,

$$f(t^{\mathcal{M}}(s)) = t^{\mathcal{N}}(s \circ f),$$

where $\circ$ is the composition operator. Conservative homomorphism f on $\mathcal{M}$ defines unambiguously the structure $\mathcal{N}$ such that $f : M \stackrel{on}{\rightarrow} N$. We denote such a structure by $f\mathcal{M}$. Structure $\mathcal{M}$ is *isomorphic* with structure $\mathcal{N}$ if and only if there is a one-to-one conservative homomorphism from $\mathcal{M}$ onto $\mathcal{N}$. Isomorphic structures are indistinguishable in the sense that thy possess the same mathematical properties. In particular, they satisfy exactly the same formulas of L. For these reasons we treat any pair of isomorphic structures as if they were equal.

If $\propto$ is a binary relation on a class $\mathbf{S}$ then a subclass $\mathbf{K}$ of $\mathbf{S}$ is called *upward* $\propto$*-closed* if and only if $\mathcal{M} \in \mathbf{K}$ and $\mathcal{M} \propto \mathcal{N}$ imply $\mathcal{N} \in \mathbf{K}$. The upward $\propto$-closure $Clo^{\uparrow}_{\propto}(\mathbf{K})$ of $\mathbf{K}$ is defined as the least upward $\propto$-closed class containing

[2]The concept of positiveness is purely syntactic, so that a non-positive formula may be logically equivalent to a positive one; similarly, a non-atomic formula may be logically equivalent to an atomic formula.

K as a subset. If $\propto$ is a reflexive and transitive relation then

$$Clo^{\uparrow}_{\propto}(\mathbf{K}) = \{\mathcal{N} \in \mathbf{S} \mid \bigvee_{\mathcal{M} \in \mathbf{K}} \mathcal{M} \propto \mathcal{N}\}. \tag{3}$$

Class **K** is called $\propto$-dense in class **J** if and only if

$$\bigwedge_{\mathcal{M} \in \mathbf{J}} \bigvee_{\mathcal{N} \in \mathbf{K} \cap \mathbf{J}} \mathcal{N} \propto \mathcal{M}. \tag{4}$$

If $\propto$ is a reflexive and transitive relation then class **K** is $\propto$-dense in class **J** if and only if $\mathbf{J} \subseteq Clo^{\uparrow}_{\propto}(\mathbf{K} \cap \mathbf{J})$. If **K** is $\propto$-dense in **J** and $\mathbf{K} \cap \mathbf{J}$ is $\propto$-dense in **I** then **K** is $\propto$-dense in **I**, also.

Transitive and reflexive closure of a relation $\propto$ is denoted by $\propto^*$. If $\propto_1$ and $\propto_2$ are binary relations on **S** then $\propto_1 \circ \propto_2$ denotes their composition, and $\propto_1 + \propto_2$ denotes their combination, that is, the relation $(\propto_1 \cup \propto_2)^*$. If $\propto_1$ and $\propto_2$ are reflexive and transitive then $\propto_1 + \propto_2 = (\propto_1 \circ \propto_2)^*$. Let us note that $\propto_1 \circ \propto_2$ is not necessarily a transitive relation, even though $\propto_1$ and $\propto_2$ are. However, if $\propto_1$, $\propto_2$ and $\propto_1 \circ \propto_2$ are reflexive and transitive then $\propto_1 + \propto_2 = \propto_1 \circ \propto_2$; the converse is true as well. We call $\propto$ a *partial order* if, in addition to being reflexive and transitive, it is *antisymmetric*, that is, it satisfies the following postulate:

$$\bigwedge_{\mathcal{N}, \mathcal{M} \in \mathbf{S}} \mathcal{N} \propto \mathcal{M} \text{ and } \mathcal{M} \propto \mathcal{N} \text{ implies } \mathcal{N} = \mathcal{M}. \tag{5}$$

3 Model Refinements

It turns out that appropriately chosen preference relations, which we call here refinements, constitute semantical counterpart of Γ-default assumption. The purpose of a refinement is to modify a model $\mathcal{M}$ of a set of sentences Σ to another model $\mathcal{N}$ of Σ, which satisfies more consequences of an instance of Γ-default assumption in question and, in effect, better approximates the reality that Σ under this default assumption is supposed to describe. Consequently, the mostly preferred models of Σ which cannot be further refined will provide $DA_{\Gamma}(\Sigma)$ with adequate semantics. In Sections 4 and 5 we will determine which refinements are appropriate matches for selected instances of the Γ-default assumption. Here, we define refinements as binary relations between first-order structures of $\mathbf{S}_L$, and state their basic properties.

A *refinement*, usually denoted by $\propto$ (possibly with subscripts), is a partial order on the class $\mathbf{S}_L$ of first-order structures for L. $\mathcal{N} \propto \mathcal{M}$ expresses the fact that $\mathcal{N}$ is a refinement of $\mathcal{M}$. Given $\mathbf{K} \subseteq \mathbf{S}_L$, the class $fin_{\propto}(\mathbf{K})$ of $\propto$-finest structures in **K** is defined by

$$fin_{\propto}(\mathbf{K}) = \{\mathcal{M} \in \mathbf{K} \mid \bigwedge_{\mathcal{N} \in \mathbf{K}} (\mathcal{N} \propto \mathcal{M} \supset \mathcal{M} = \mathcal{N})\}. \tag{6}$$

The entailment $\vdash$ is incomplete with respect to the $\propto$-finest semantics. Indeed, a sentence φ that is not provable from Σ by means of $\vdash$, or in other words, is false in some elements of $\mathbf{Mod}(\Sigma)$, may be true in all elements of $fin_{\propto}(\mathbf{Mod}(\Sigma))$. This fact motivates our interest in the entailment $\vdash_{fin_{\propto}}$, denoted for simplicity by $\vdash_{\propto}$, which by definition (2) *is* complete with respect to the $\propto$-finest semantics. Since $fin_{\propto}(\mathbf{K})$ always is a subset of $\mathbf{K}$, $\vdash_{\propto}$ is characterized by the following instance of (2):

$$\Sigma \vdash_{\propto} \varphi \text{ iff } fin_{\propto}(\mathbf{Mod}(\Sigma)) \subseteq \mathbf{Mod}(\varphi). \tag{7}$$

$\Sigma \vdash_{\propto} \varphi$ means that φ is true in all $\propto$-finest models of Σ. We call $\vdash_{\propto}$ a refined entailment. Of course, if $\Sigma \vdash \varphi$ then $\Sigma \vdash_{\propto} \varphi$, but the converse is not necessarily true.

A subclass $\mathbf{K}$ of $\mathbf{S}_L$ is called $\propto$-*flat* if and only if,

$$\bigwedge_{\mathcal{M},\mathcal{N} \in \mathbf{K}} \mathcal{M} \propto \mathcal{N} \text{ implies } \mathcal{M} = \mathcal{N}. \tag{8}$$

Obviously, by (6), every class of the form $fin_{\propto}(\mathbf{K})$ is flat, but not vice versa.

We focus on the following instances of the refinement $\propto$. It is a matter of routine inspection to check that all these instances are partial orders.

3.1 Predicate Minimization

The first instance of $\propto$ is the relation $\leq$ of enlargement (introduced in [12]) in the class $\mathbf{S}_L$ of first-order structures for L. It is defined as follows:

> $(\mathcal{M} \leq \mathcal{N})$ if and only if ($M = N$, and for every function symbol F of L, $F^{\mathcal{M}} = F^{\mathcal{N}}$, and for every predicate symbol P not in $\Re$, $P^{\mathcal{M}} = P^{\mathcal{N}}$, and for every predicate symbol R in $\Re$, $R^{\mathcal{M}} \subseteq R^{\mathcal{N}}$).

Numerous variants of $\leq$-finest semantics turned out to be particularly useful in deductive data bases, logic programming, and other areas of artificial intelligence. One of the applications of this semantics is the definition of the *transitive closure* TC_R of a binary relation R. Although TC_R is generally not first-order definable, the following sentence defines TC_R in its $\leq$-finest models: $(\forall x)(\forall y)(R(x,y) \supset TC_R(x,y)) \wedge (\forall x)(\forall y)(\forall z)((TC_R(x,y) \wedge TC_R(y,z)) \supset R'(x,z))$, where the only predicate minimized by $\leq$ is TC_R. Consequently, using minimization one can define relational semantics of programs.

Following generally accepted convention, we call $\leq$-finest models *predicate-minimal models*.

3.2 Domain Minimization

The very relation $\subseteq$ between first-order structures comprises our second instance of refinement. It formalizes the *Ockham's razor principle* in this sense, that the domains of $\subseteq$-finest structures consist only of necessary individua, namely of interpretations of *constant*[3] terms of the language L. We call $\subseteq$-finest structures *domain-minimal structures*. Not every consistent set of sentences has a model being a domain-minimal structure. For example, $\{\exists x P(x)\} \cup \{\neg P(t) \mid t$ is a constant term of $L\}$ has not. However, for every theory $\Sigma \subseteq \forall$, each domain-minimal substructure of a model of Σ is a model of Σ.

Domain-minimal structures are also known under the name *domain-closed structures*.

3.3 Gluing

As an alternative for $\subseteq$, one may consider the relation $\trianglelefteq$ of projection (introduced by Lyndon in [12])defined by:

> $\mathcal{M} \trianglelefteq \mathcal{N}$ **iff** there exists a conservative homomorphism f such that $\mathcal{M} = f\mathcal{N}$.

The nature of the refinement $\trianglelefteq$ lies in gluing together certain elements of the domain of a model if they satisfy the same atomic and negated atomic sentences of L. Therefore, $\trianglelefteq$ may be used to minimize the domains of first-order structures, albeit, in a different sense than $\subseteq$. For instance, in a language with one unary predicate P and three constants a, b, and c, each $\trianglelefteq$-finest model of $\{P(a), \neg P(b), \neg P(c)\}$ has the domain consisting of two elements: one to interpret a and another to interpret b and c.

3.4 Discrimination

Oddly enough, something quite converse to gluing $\trianglelefteq$, namely *discrimination*, seems to be a matter of interest in deductive databases and logic programming. Discrimination may be defined as the inverse of gluing restricted to elements of the domain which are values of the constant terms of L. We denote the relation of discrimination by $\trianglerighteq$. $\mathcal{M} \trianglerighteq \mathcal{N}$ means that there is a conservative homomorphism f of $\mathcal{M}$ onto $\mathcal{N}$, such that $f(m_1) = f(m_2)$ holds only if, for some constant terms t_1 and t_2 of L, $m_1 = t_1^{\mathcal{M}}$, and $m_2 = t_2^{\mathcal{M}}$. In particular,

$$\mathcal{M} \trianglerighteq \mathcal{N} \text{ implies } \bigvee_{\mathcal{A} \subseteq \mathcal{M}} \mathcal{A} \trianglerighteq \mathcal{M}. \tag{9}$$

Models which are $\trianglerighteq$-finest are called *discriminant*, or, in the extreme case, when every two non-identical constant terms are assigned different values, they are

[3] i.e., with no occurrences of variables

called *strongly discriminant*. For instance, if f is a function symbol and c is a constant of L then each strongly discriminant structure satisfies $\neg(f(c) = c)$. Discriminant models constitute a semantic counterpart of the so called *unique-names assumption*.

The restriction to strongly discriminant semantics is necessary for the validity of many published completeness proofs of *negation-as-failure* and similar procedures. This is most likely the reason of popularity of discriminant models. Of course, if Σ contains occurrences of the equality symbol, for instance, a sentence $a = b$, there may be no strongly discriminant models of Σ. However, we will show in subsection 3.6 that for every consistent $\Sigma \subseteq \forall$, there are always discriminant models of Σ, which are strongly discriminant if $=$ does not occur in Σ.

3.5 Combinations

It turns out that the most interesting in applications are certain combinations of the above instances of $\propto$. For example, the combination of $\subseteq$ and $\trianglerighteq$ leads to the concept of the *Herbrand* (i.e., domain-minimal and discriminant) *structure*, or, in the extreme case to *strongly Herbrand* (i.e., domain-minimal and strongly discriminant) *structure*. The combination of the three refinements: $\subseteq$, $\trianglerighteq$, and $\leq$ yields in the extreme case the concept of a *minimal strongly Herbrand structure*. This notion plays a fundamental role in model theory for deductive databases and logic programming.

Combinations of refinements are the central point of our interest. They are the subject of systematic study in Section 5.

3.6 Densly Founded Theories

If certain theories have models which cannot be refined to $\propto$-finest models than the entailment $\vdash_{\propto}$ may reveal an irregular behavior. In particular, if a consistent theory Σ has no $\propto$-finest models at all (e.g., in the language with a unary predicate symbol N, the usual inductive definition of natural numbers with the postulate "$N(0)$" being replaced by "$\exists x N(x)$", has no predicate-minimal models) then Σ entails *false*, which is decidedly an unwanted consequence. This troubled situation cannot happen if every $\Sigma \subseteq \forall$ is $\propto$-densly founded, that is, if class $fin_{\propto}(\mathbf{Mod}(\Sigma))$ is $\propto$-dense in $\mathbf{Mod}(\Sigma)$. Fortunately, this is the case for all the refinements considered in this paper.

Lemma 3.6.1 For every $\Sigma \subseteq \forall$,

i. $fin_{\leq}(\mathbf{Mod}(\Sigma))$ is $\leq$-dense in $\mathbf{Mod}(\Sigma)$.

ii. $fin_{\subseteq}(\mathbf{Mod}(\Sigma))$ is $\subseteq$-dense in $\mathbf{Mod}(\Sigma)$.

iii. $fin_{\trianglelefteq}(\mathbf{Mod}(\Sigma))$ is $\trianglelefteq$-dense in $\mathbf{Mod}(\Sigma)$.

iv. $fin_{\trianglelefteq}(\mathbf{Mod}(\Sigma))$ is $\trianglerighteq$-dense in $\mathbf{Mod}(\Sigma)$.

Proof. (i) was proven in [2]. (ii). Because every first-order structure contains a domain-minimal structure as a substructure, the density of $fin_{\subseteq}(\mathbf{Mod}(\Sigma))$ in $\mathbf{Mod}(\Sigma)$ is secured by the Łoś–Tarski Theorem (see [8], thm 3.11). (iii) was proven in [22], lem. 7.4. (iv). Let $\mathbf{K} = \{\mathcal{N} \trianglerighteq \mathcal{A} \mid \mathcal{N} \models \Sigma\}$, and let $\mathcal{A}_1 \trianglelefteq ... \trianglelefteq \mathcal{A}_i \trianglelefteq ...$, $i \in I$, be a chain in $\mathbf{K}$. Let $E(\mathcal{M})$ denote the set of all equational atomic sentences of L satisfied by $\mathcal{M}$. There exists $\mathcal{B} \in \mathbf{S}_L$, with $\mathcal{B} \trianglerighteq \mathcal{A}$ and $E(\mathcal{B}) = \bigcap_{i \in I} E(\mathcal{A}_i)$. Of course, $\mathcal{B} \trianglerighteq \mathcal{A}_i$ for each $i \in I$. Let f_i be a conservative homomorphism from $\mathcal{B}$ onto $\mathcal{A}_i$. An inductive argument shows that for every quantifier-free formula φ of L, and every assignment s in B,

$$\bigvee_{i \in I} \bigwedge_{j,k \in I : j,k > i} \mathcal{A}_j \models \varphi[s \circ f_j] \text{ iff } \mathcal{A}_k \models \varphi[s \circ f_k]$$

(the case of atomic and negated atomic equational formula follows from the definition of $\mathcal{B}$, the case of atomic and negated atomic predicate formula follows from the conservativeness of f_j, the cases of $\varphi_1 \vee \varphi_2$ and $\varphi_1 \wedge \varphi_2$ are routine). From this we conclude, by induction again,

$$\mathcal{B} \models \varphi[s] \text{ iff } \bigvee_{i \in I} \bigwedge_{j \in I : i < j} \mathcal{A}_j \models \varphi[s \circ f_j]$$

Hence, for every sentence $\varphi \in \forall$,

$$\mathcal{B} \models \varphi \text{ iff } \bigvee_{i \in I} \bigwedge_{j \in I : i < j} \mathcal{A}_j \models \varphi.$$

Since $\bigwedge_{i \in I} \mathcal{A}_i \models \Sigma$, we conclude $\mathcal{B} \models \Sigma$. By the Kuratowski - Zorn Lemma (cf. [6], thm 4.1), $\mathbf{K}$ contains a minimal (with respect to $\trianglerighteq$) element $\mathcal{M}$. Obviously, $\mathcal{M} \trianglerighteq \mathcal{A}$ and $\mathcal{M} \in fin_{\trianglelefteq}(\mathbf{Mod}(\Sigma))$. Hence (iv). □

Lemma 3.6.1 is surprisingly strong. For instance, iv implies the compactness theorem of $\forall$-fragment of first-order logic. In Section 5 we will generalize Lemma 3.6.1 over combinations of refinements.

4 Simple Preservation Properties

Let Γ be a set of $\forall$-sentences of L. The Γ-default assumption DA_Γ introduced in Section 1 is formally defined as a consequence operation from subsets of $\forall$ into subsets of $\forall$, given by:

$$\bigwedge_{\Sigma \subseteq \forall} \bigwedge_{\varphi \in L} \varphi \in DA_\Gamma(\Sigma) \text{ iff } \varphi \in \forall \text{ and } (\Sigma \cup \{\varphi\})_\Gamma = \Sigma_\Gamma. \tag{10}$$

DA_Γ assumes, by default, a clause φ if and only if φ does not introduce new Γ-consequences to Σ. For $\varphi \in \forall$, scheme (10) generalizes (1) over infinite Γ's; one can check that if Γ is finite then a working instance of (1) can be obtained by putting $\Gamma_0 = \Sigma_\Gamma$.

It turns out that those Γ's, whose elements' truthfulness is upward $\propto$-preserved, yield instances of Γ-default assumption which are $\forall$-complete with respect to the refined entailments $\vdash_\propto$.

Definition 4.1 We call a set Γ of sentences of L a complete preservation property with respect to the refinement $\propto$, in abbreviation: CPP(Γ, α), if and only if,

$$\bigwedge_{\Sigma \subseteq \forall} \mathbf{Mod}(\Sigma_\Gamma) = Clo^{\uparrow}_{\propto}(\mathbf{Mod}(\Sigma)). \tag{11}$$

□

In other words, CPP$(\Gamma, \propto)$ means that for every $\Sigma \subseteq \forall$, $\mathbf{Mod}(\Sigma_\Gamma)$ is the least upward $\propto$-closed class containing $\mathbf{Mod}(\Sigma)$ as a subset. Because we assumed that refinements are partial orders, in particular reflexive and transitive, CPP$(\Gamma, \propto)$ holds if and only if for every $\Sigma \subseteq \forall$ and every $\mathcal{A} \in \mathbf{S}_\mathcal{L}$,

$$\mathcal{A} \models \Sigma_\Gamma \text{ iff } \bigvee_{\mathcal{M} \propto \mathcal{A}} \mathcal{M} \models \Sigma. \tag{12}$$

CPP$(\Gamma, \propto)$ implies that any $\forall$-axiomatizable subclass of $\mathbf{S}_L$ is upward $\propto$-closed if and only if it is axiomatizable by a subset of Γ. In particular, for every $\varphi \in \Gamma$, if $\mathcal{M} \models \varphi$ and $\mathcal{M} \propto \mathcal{N}$ then $\mathcal{N} \models \varphi$. Thus $\propto$ preserves the truthfulness of sentences of Γ. Moreover, for every sentence φ of L whose truthfulness is preserved under $\propto$ there is a finite $\Gamma_0 \subseteq \Gamma$ with $\vdash \varphi \equiv \wedge\Gamma_0$. In this sense Γ is a *complete* preservation property.

The notion of complete preservation property constitutes a fundamental concept which allows us to establish the $\forall$-completeness of the Γ-default assumption with respect to refined entailment $\vdash_\propto$.

The Completeness Theorem 4.2 Assume that CPP$(\Gamma, \propto)$ holds. For every $\propto$-densly founded $\Sigma \subseteq \forall$ and $\varphi \in \forall$,

$$\Sigma \vdash_\propto \varphi \text{ iff } \varphi \in DA_\Gamma(\Sigma).$$

Proof. We show that under these assumptions, $\Sigma \vdash_\propto \varphi$ if and only if $(\Sigma \cup \{\varphi\})_\Gamma = \Sigma_\Gamma$. Indeed, $\Sigma \vdash_\propto \varphi$ iff [by definition (7) of $\vdash_\propto$] $fin_\propto(\mathbf{Mod}(\Sigma)) \subseteq \mathbf{Mod}(\varphi)$ iff $\mathbf{Mod}(\varphi)$ is $\propto$-dense in $fin_\propto(\mathbf{Mod}(\Sigma))$ iff [by $\propto$-densly foundedness of Σ, that is by $\propto$-density of $fin_\propto(\mathbf{Mod}(\Sigma))$ in $\mathbf{Mod}(\Sigma)$] $\mathbf{Mod}(\varphi)$ is $\propto$-dense in $\mathbf{Mod}(\Sigma)$ iff [by (3) and reflexiveness and transitivity of $\propto$] $\mathbf{Mod}(\Sigma) \subseteq Clo^{\uparrow}_{\propto}(\mathbf{Mod}(\Sigma) \cap \mathbf{Mod}(\varphi))$ iff $\mathbf{Mod}(\Sigma) \subseteq Clo^{\uparrow}_{\propto}(\mathbf{Mod}(\Sigma \cup \{\varphi\}))$ iff [by CPP$(\Gamma, \propto)$ and (11)] $\mathbf{Mod}(\Sigma) \subseteq \mathbf{Mod}((\Sigma \cup \{\varphi\})_\Gamma)$. Observation that $\Sigma_\Gamma \subseteq (\Sigma \cup \{\varphi\})_\Gamma$ always holds completes the proof. □

We demonstrated in Section 3 (Lemma 3.6.1) that all the instances of $\propto$ discussed there satisfied the requirement that $\forall$-theories are $\propto$-densly founded. Here, we establish their complete preservation properties.

Lemma 4.3 The following complete preservation properties hold:

i. $CPP(Pos_{\{=\}'}, \leq)$,

ii. $CPP(QF, \subseteq)$,

iii. $CPP(Neg_{\{=\}}, \trianglelefteq)$,

iv. $CPP(Pos_{\{=\}}, \trianglerighteq)$,

Proof. (i) We have to demonstrate that

$$\bigwedge_{\Sigma \subseteq \forall} \mathbf{Mod}(\Sigma_{Pos_{\{=\}'}}) = Clo^{\uparrow}_{\leq}(\mathbf{Mod}(\Sigma)).$$

Because $\leq$ is a transitive and reflexive relation, it suffices to show that for every $\Sigma \subseteq \forall$ and every $\mathcal{A} \in \mathbf{S}_L$,

$$\mathcal{A} \models \Sigma_{Pos_{\{=\}'}} \text{ iff } \bigvee_{\mathcal{M} \leq \mathcal{A}} \mathcal{M} \models \Sigma.$$

This has been done in [22], cor. 3.6.

(ii) We have to demonstrate that

$$\bigwedge_{\Sigma \subseteq \forall} \mathbf{Mod}(\Sigma_{QF}) = Clo^{\uparrow}_{\subseteq}(\mathbf{Mod}(\Sigma)).$$

Because $\subseteq$ is a transitive relation, it suffices to show that for every $\Sigma \subseteq \forall$ and every $\mathcal{A} \in \mathbf{S}_L$,

$$\mathcal{A} \models \Sigma_{QF} \text{ iff } \bigvee_{\mathcal{M} \subseteq \mathcal{A}} \mathcal{M} \models \Sigma.$$

($\Leftarrow$) Assume $\mathcal{A} \not\models \Sigma_{QF}$. Take any $\varphi \in \Sigma_{QF}$ with $\mathcal{A} \not\models \varphi$, that is, $\mathcal{A} \models \neg\varphi$, and any $\mathcal{M} \subseteq \mathcal{A}$. Of course, $\mathcal{M} \models \neg\varphi$, that is, $\mathcal{M} \not\models \Sigma_{QF}$.
($\Rightarrow$) Let $\mathcal{A} \models \Sigma_{QF}$ and let $\mathcal{B}$ be a domain-minimal substructure of $\mathcal{A}$. We show that $\mathcal{B} \models \Sigma$. Suppose not, that is, let $\varphi \in \Sigma$ be such that $\mathcal{B} \not\models \varphi$. Because $\Sigma \subseteq \forall$, φ has the form of $\forall\vec{x}\psi(\vec{x})$, where $\psi(\vec{x})$ is quantifier-free. We have $\mathcal{B} \models \exists\vec{x}\neg\psi(\vec{x})$. Since B is a domain-minimal structure, there exists a sequence $\vec{c}$ of constant terms of L with $\mathcal{B} \models \neg\psi(\vec{c})$. Because $\mathcal{B} \subseteq \mathcal{A}$, we obtain $\mathcal{A} \models \neg\psi(\vec{c})$. On the other hand, $\psi(\vec{c}) \in \Sigma_{QF}$, therefore $\mathcal{A} \models \psi(\vec{c})$, a contradiction.

(iii) We have to demonstrate that

$$\bigwedge_{\Sigma \subseteq \forall} \mathbf{Mod}(\Sigma_{Neg_{\{=\}}}) = Clo^{\uparrow}_{\trianglelefteq}(\mathbf{Mod}(\Sigma)).$$

Because $\trianglelefteq$ is a transitive and reflexive relation, it suffices to show that for every $\Sigma \subseteq \forall$ and every $\mathcal{A} \in \mathbf{S}_L$,

$$\mathcal{A} \models \Sigma_{Neg_{\{=\}}} \text{ iff } \bigvee_{\mathcal{M} \not\trianglelefteq \mathcal{A}} \mathcal{M} \models \Sigma.$$

This has been done in [22], cor. 7.2.

(iv) Obviously, $Clo^{\uparrow}_{\trianglelefteq}(\mathbf{Mod}(\Sigma)) \subseteq Clo^{\uparrow}_{\trianglelefteq}(\mathbf{Mod}(\Sigma))$. $Clo^{\uparrow}_{\trianglelefteq}(\mathbf{Mod}(\Sigma)) \subseteq$ [by (9)] $Clo^{\uparrow}_{\trianglelefteq}(Clo^{\uparrow}_{\subseteq}(\mathbf{Mod}(\Sigma))) = Clo^{\uparrow}_{\trianglelefteq}(\mathbf{Mod}(\Sigma))$. The proof of

$$\bigwedge_{\Sigma \subseteq \forall} \mathbf{Mod}(\Sigma_{Pos_{\{=\}}}) = Clo^{\uparrow}_{\trianglelefteq}(\mathbf{Mod}(\Sigma))$$

is symmetric to (iii). Hence, (iv).

□

Now, Theorem 4.2 yields automatically the appropriate $\forall$-completeness results.

Theorem 4.4 For every $\Sigma \subseteq \forall$ and every $\varphi \in \forall$,

i. $\Sigma \vdash_{\leq} \varphi$ iff $\varphi \in DA_{Pos_{\{=\}'}}(\Sigma)$

ii. $\Sigma \vdash_{\subseteq} \varphi$ iff $\varphi \in DA_{QF}(\Sigma)$

iii. $\Sigma \vdash_{\not\trianglelefteq} \varphi$ iff $\varphi \in DA_{Neg_{\{=\}}}(\Sigma)$

iv. $\Sigma \vdash_{\trianglelefteq} \varphi$ iff $\varphi \in DA_{Pos_{\{=\}}}(\Sigma)$

Proof by application of Lemmas 3.6.1 and 4.3, and Theorem 4.2 □

The above theorem makes it clear that the $Pos_{\{=\}'}$-default assumption expresses the *predicate circumscription*, the QF-default assumption expresses the *domain closure axiom*, and the $Pos_{\{=\}}$-default assumption expresses the *unique-names assumption*.

$DA_{Pos_{\{=\}'}}$ was first introduced as an instance of operation cwa_S, and proven complete with respect to entailment $\vdash_{\leq}$ in [21].

5 Composed Preservation Properties

In this section, we investigate combinations of refinements. We prove that the resulting $\forall$-complete default assumptions are straightforward combinations of the simple ones.

We start with technical lemmas (Lemmas 5.1 - 5.8) which yield a handy tool for surgically clean analysis of the following combinations of instances of $\propto$ that were introduced in Sections 3 and 4:

- simultaneous predicate minimization and domain minimization (we call it a total minimization),
- predicate minimization with discrimination,
- domain minimization with discrimination, and
- total minimization with discrimination.

Lemma 5.1 Assume that $\propto_1 \circ \propto_2$ is transitive. Then for every $\mathbf{K} \subseteq \mathbf{S}_L$,

$$Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{K})) = Clo^{\uparrow}_{\propto_1}(Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{K}))). \tag{13}$$

Proof. It suffices to show that $Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{K}))$ is upward $\propto_1$-closed. Let $\mathcal{M} \in Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{K}))$, and $\mathcal{M} \propto_1 \mathcal{N}$. By the reflexiveness of $\propto_2$ we get $\mathcal{M} \propto_1 \circ \propto_2 \mathcal{N}$. By (3) we obtain $\bigvee_{\mathcal{A},\mathcal{B}\in\mathbf{K}} \mathcal{A} \propto_1 \mathcal{B} \propto_2 \mathcal{M}$, that is, $\bigvee_{\mathcal{A}\in\mathbf{K}} \mathcal{A} \propto_1 \circ \propto_2 \mathcal{M}$. The transitivity of $\propto_1 \circ \propto_2$ yields $\bigvee_{\mathcal{A}\in\mathbf{K}} \mathcal{A} \propto_1 \circ \propto_2 \mathcal{N}$, that is, by (3) again, $\mathcal{N} \in Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{K}))$. Thus $Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{K}))$ is upward $\propto_1$-closed. □

We say that sets Γ_1 and Γ_2 possess the *intersection property* if and only if every subset of Γ_1 logically equivalent to a subset of Γ_2 is logically equivalent to a subset of $\Gamma_1 \cap \Gamma_2$. For instance, one can show that the sets $\forall$ and $\exists$ possess the intersection property (note that $\forall \cap \exists = QF$).

The following lemma is interesting in its own right since it generalizes classic results of Craig [3] and Lyndon [11].

Interpolation Lemma 5.2 Assume that $\mathrm{CPP}(\Gamma_1, \propto_1)$ and $\mathrm{CPP}(\Gamma_2, \propto_2)$ hold, $\propto_1 \circ \propto_2$ is a partial order, and Γ_1 and Γ_2 posses the intersection property. Then

$$\bigwedge_{\Sigma\subseteq\forall} \mathbf{Mod}(\Sigma_{\Gamma_1\cap\Gamma_2}) = \mathbf{Mod}((\Sigma_{\Gamma_1})_{\Gamma_2}). \tag{14}$$

Proof. $\Sigma_{\Gamma_1\cap\Gamma_2} = Cn(\Sigma) \cap \Gamma_1 \cap \Gamma_2 \subseteq Cn(Cn(\Sigma) \cap \Gamma_1) \cap \Gamma_2 = (\Sigma_{\Gamma_1})_{\Gamma_2}$. Hence, $\mathbf{Mod}((\Sigma_{\Gamma_1})_{\Gamma_2}) \subseteq \mathbf{Mod}(\Sigma_{\Gamma_1\cap\Gamma_2})$. Therefore, it suffices to prove that $\mathbf{Mod}(\Sigma_{\Gamma_1\cap\Gamma_2}) \subseteq \mathbf{Mod}((\Sigma_{\Gamma_1})_{\Gamma_2})$. Indeed, $\mathbf{Mod}((\Sigma_{\Gamma_1})_{\Gamma_2})$ = [by $\mathrm{CPP}(\Gamma_2, \propto_2)$ and (3)] $Clo^{\uparrow}_{\propto_2}(\mathbf{Mod}(\Sigma_{\Gamma_1}))$ = [by $\mathrm{CPP}(\Gamma_1, \propto_1)$ and (3)] $Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{Mod}(\Sigma)))$ = [by Lemma 5.1] $Clo^{\uparrow}_{\propto_1}(Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{Mod}(\Sigma))))$ = [by similar argument] $\mathbf{Mod}(((\Sigma_{\Gamma_1})_{\Gamma_2})_{\Gamma_1})$. Because $((\Sigma_{\Gamma_1})_{\Gamma_2})_{\Gamma_1} \subseteq \Gamma_1$ and $(\Sigma_{\Gamma_1})_{\Gamma_2} \subseteq \Gamma_1$, the intersection property of Γ_1 and Γ_2 yields $\Pi \subseteq \Gamma_1 \cap \Gamma_2$ with $\mathbf{Mod}((\Sigma_{\Gamma_1})_{\Gamma_2}) = \mathbf{Mod}(\Pi)$. Since $\mathbf{Mod}(\Sigma) \subseteq \mathbf{Mod}((\Sigma_{\Gamma_1})_{\Gamma_2})$, we have $\mathbf{Mod}(\Sigma) \subseteq \mathbf{Mod}(\Pi)$, therefore $\Pi \subseteq \Sigma_{\Gamma_1\cap\Gamma_2}$. Hence, $\mathbf{Mod}(\Sigma_{\Gamma_1\cap\Gamma_2}) \subseteq \mathbf{Mod}(\Pi)$. Thus $\mathbf{Mod}(\Sigma_{\Gamma_1\cap\Gamma_2}) \subseteq \mathbf{Mod}((\Sigma_{\Gamma_1})_{\Gamma_2})$. □

To see the "interpolation" in Lemma 5.2 let us note that (14) can be equivalently expressed by:

$$\bigwedge_{\Pi\subseteq\Gamma_1} \bigwedge_{\varphi\in\Gamma_2} (\Pi \vdash \varphi \text{ implies } \bigvee_{\Theta\subseteq\Gamma_1\cap\Gamma_2} (\Pi \vdash \Theta \text{ and } \Theta \vdash \varphi)),$$

where $\Pi \vdash \Theta$ is an abbreviation for $\bigwedge_{\vartheta \in \Theta} \Pi \vdash \vartheta$.

Now, we prove that CPP is preserved under combinations of refinements.

Lemma 5.3 Assume that CPP$(\Gamma_1, \propto_1)$ and CPP$(\Gamma_2, \propto_2)$ hold, $\propto_1 \circ \propto_2$ is a partial order, and Γ_1 and Γ_2 have the intersection property. Then CPP$(\Gamma_1 \cap \Gamma_2, \propto_1 + \propto_2)$ holds.

Proof. We must prove that

$$\bigwedge_{\Sigma \subseteq \forall} \mathbf{Mod}(\Sigma_{\Gamma_1 \cap \Gamma_2}) = Clo^{\uparrow}_{\propto_1 + \propto_2}(\mathbf{Mod}(\Sigma)). \tag{15}$$

Because $\propto_1 + \propto_2$ is reflexive and transitive, it suffices to demonstrate that for every $\Sigma \subseteq \forall$ and $\mathcal{A} \in \mathbf{S}_L$,

$$\mathcal{A} \models \Sigma_{\Gamma_1 \cap \Gamma_2} \text{ iff } \bigvee_{\mathcal{M} \propto_1 + \propto_2 \mathcal{A}} \mathcal{M} \models \Sigma. \tag{16}$$

By Lemma 5.2, $\mathbf{Mod}(\Sigma_{\Gamma_1 \cap \Gamma_2}) = \mathbf{Mod}((\Sigma_{\Gamma_1})_{\Gamma_2})$. Therefore, $\mathcal{A} \models \Sigma_{\Gamma_1 \cap \Gamma_2}$ iff $\mathcal{A} \models (\Sigma_{\Gamma_1})_{\Gamma_2}$ iff [because of $CPP(\Gamma_1, \propto_1$, $CPP(\Gamma_2, \propto_2$, and (3)] $\mathcal{A} \in Clo^{\uparrow}_{\propto_2}(Clo^{\uparrow}_{\propto_1}(\mathbf{Mod}(\Sigma)))$ iff [because of reflexiveness and transitivity of $\propto_2$] $\bigvee_{\mathcal{N} \propto_2 \mathcal{A}} \mathcal{N} \in Clo^{\uparrow}_{\propto_1}(\mathbf{Mod}(\Sigma))$ iff [because of transitivity of $\propto_1$] $\bigvee_{\mathcal{N} \propto_2 \mathcal{A}} \bigvee_{\mathcal{M} \propto_1 \mathcal{N}} \mathcal{M} \models \Sigma$ iff $\bigvee_{\mathcal{M} \propto_1 \circ \propto_2 \mathcal{A}} \mathcal{M} \models \Sigma$ iff [because of transitivity of $\propto_1 \circ \propto_2$, $\propto_1 + \propto_2 = \propto_1 \circ \propto_2$] $\bigvee_{\mathcal{M} \propto_1 + \propto_2 \mathcal{A}} \mathcal{M} \models \Sigma$. Hence (16). □

Here are the rest of the technicalities.

Lemma 5.4 For every upward $\propto_1 + \propto_2$-closed class $\mathbf{K} \subseteq \mathbf{S}_L$,

$$fin_{\propto_1 + \propto_2}(\mathbf{K}) = fin_{\propto_1}(\mathbf{K}) \cap fin_{\propto_2}(\mathbf{K}). \tag{17}$$

Proof. The case of $(\subseteq)$ follows from $fin_{\propto}(\mathbf{K}) \subseteq \mathbf{K}$.

$(\supseteq)$ Assume $\mathcal{M} \in fin_{\propto_1}(\mathbf{K}) \cap fin_{\propto_2}(\mathbf{K})$. Let $\mathcal{N} \in \mathbf{K}$ satisfy $\mathcal{N} \propto_1 + \propto_2 \mathcal{M}$, with $\mathcal{N} \neq \mathcal{M}$. There is $\mathcal{A} \in \mathbf{S}_L$, $\mathcal{A} \neq \mathcal{M}$, with $\mathcal{N} \propto_1 + \propto_2 \mathcal{A} \propto_i \mathcal{M}$, where $i \in \{1, 2\}$. By upward $\propto_1 + \propto_2$-closedness of $\mathbf{K}$, $\mathcal{A} \in \mathbf{K}$. Hence, $\mathcal{M} \notin fin_{\propto_i}(\mathbf{K})$, a contradiction. □

Lemma 5.5 For every $\mathbf{K} \subseteq \mathbf{S}_L$,

$$fin_{\propto}(\mathbf{K}) = fin_{\propto}(Clo^{\uparrow}_{\propto}(\mathbf{K})). \tag{18}$$

Proof. $(\subseteq)$ Assume $\mathcal{M} \in fin_{\propto}(\mathbf{K})$ and $\mathcal{N} \in Clo^{\uparrow}_{\propto}(\mathbf{K})$, with $\mathcal{N} \propto \mathcal{M}$. There is $\mathcal{A} \in \mathbf{K}$ with $\mathcal{A} \propto \mathcal{N}$. The transitivity of $\mathcal{M}$ yields $\mathcal{A} \propto \mathcal{M}$. $\propto$-finestness of $\mathcal{M}$ gives $\mathcal{A} = \mathcal{M}$. Therefore, both $\mathcal{N} \propto \mathcal{M}$ and $\mathcal{M} \propto \mathcal{N}$ hold. Because $\propto$ is a partial order, (5) yields $\mathcal{N} = \mathcal{M}$. Hence, $\mathcal{M} \in fin_{\propto}(Clo^{\uparrow}_{\propto}(\mathbf{K}))$.

$(\supseteq)$ Assume $\mathcal{M} \in fin_{\propto}(Clo^{\uparrow}_{\propto}(\mathbf{K}))$ and $\mathcal{N} \in \mathbf{K}$, with $\mathcal{N} \propto \mathcal{M}$. In particular, $\mathcal{N} \in Clo^{\uparrow}_{\propto}(\mathbf{K})$, therefore $\propto$-finestness of $\mathcal{M}$ yields $\mathcal{N} = \mathcal{M}$. Hence, $\mathcal{M} \in \mathbf{K}$, and $\mathcal{M} \in fin_{\propto}(\mathbf{K})$. □

Lemma 5.6 For every $\mathbf{K} \subseteq \mathbf{S}_L$, $fin_\propto(\mathbf{K})$ is $\propto$-dense in $\mathbf{K}$ iff $fin_\propto(\mathbf{K})$ is $\propto$-dense in $Clo^\uparrow_\propto(\mathbf{K})$.

Proof follows from $fin_\propto(\mathbf{K}) \subseteq \mathbf{K} \subseteq Clo^\uparrow_\propto(\mathbf{K})$ and $\propto$-density of $\mathbf{K}$ in $Clo^\uparrow_\propto(\mathbf{K})$. □

Lemma 5.7 For every $\mathbf{K} \subset \mathbf{S}_L$, $fin_\propto(\mathbf{K})$ is $\propto$-dense in $\mathbf{K}$ iff $fin_\propto(Clo^\uparrow_\propto(\mathbf{K}))$ is $\propto$-dense in $Clo^\uparrow_\propto(\mathbf{K})$.

Proof by direct application of Lemmas 5.5 and 5.6. □

To show that the dense foundedness of $\forall$-theories is preserved under combinations of refinements we need the following property of antipreservation. Refinement $\propto_2$ is said to be *antipreserving* for refinement $\propto_1$ if and only if

$$\bigwedge_{\mathcal{A},\mathcal{B},\mathcal{C}\in\mathbf{S}_L} \mathcal{A} \propto_2 \mathcal{B} \propto_1 \mathcal{C} \text{ and } \mathcal{A} \propto_1 \mathcal{C} \text{ imply } \mathcal{A} = \mathcal{B}. \tag{19}$$

(To see the above concept from a better perspective, one can equivalently start from this definition: $\propto_2$ is *preserving* for $\propto_1$ iff $\propto_2 \circ \propto_1 \subseteq \propto_1$, that is,

$$\bigwedge_{\mathcal{A},\mathcal{B},\mathcal{C}\in\mathbf{S}_L} \mathcal{A} \propto_2 \mathcal{B} \propto_1 \mathcal{C} \text{ implies } \mathcal{A} \propto_1 \mathcal{C},$$

and then use the prefix "anti", as in the definition (5) of "antisymmetric".)

Lemma 5.8 Assume that $\propto_2$ is antipreserving for $\propto_1$, $\propto_1 \circ \propto_2$ is a partial order, $fin_{\propto_1}(\mathbf{K})$ is $\propto_1$-dense in $\mathbf{K}$, and $fin_{\propto_2}(\mathbf{K})$ is $\propto_2$-dense in $\mathbf{K}$. Then $fin_{\propto_1+\propto_2}(\mathbf{K})$ is $\propto_1 + \propto_2$-dense in $\mathbf{K}$.

Proof. By Lemma 5.7, we may assume that $\mathbf{K}$ is upward $\propto_1 + \propto_2$-closed. Therefore, by Lemma 5.4, it suffices to show that

$$fin_{\propto_1}(\mathbf{K}) \cap fin_{\propto_2}(\mathbf{K}) \text{ is } \propto_1 + \propto_2 \text{-dense in } \mathbf{K}. \tag{20}$$

Let $\mathcal{M} \in \mathbf{K}$, $\mathcal{A} \in fin_{\propto_2}(\mathbf{K})$ with $\mathcal{A} \propto_2 \mathcal{M}$ [it exists since $fin_{\propto_2}(\mathbf{K})$ is $\propto_2$-dense in K], $\mathcal{B} \in fin_{\propto_1}(\mathbf{K})$ with $\mathcal{B} \propto_1 \mathcal{A}$ [it exists because $fin_{\propto_1}(\mathbf{K})$ is $\propto_1$-dense in K], and $\mathcal{N} \in \mathbf{K}$ with $\mathcal{N} \propto_2 \mathcal{B}$. Because $\propto_1 + \propto_2 = \propto_1 \circ \propto_2$, $\mathcal{N} \propto_2 \mathcal{B} \propto_1 \mathcal{A}$ yields $\mathcal{C} \in \mathbf{S}_L$, with $\mathcal{N} \propto_1 \mathcal{C} \propto_2 \mathcal{A}$. Since $\mathbf{K}$ is upward $\propto_1$-closed and $\mathcal{N} \in \mathbf{K}$, we see that $\mathcal{C} \in \mathbf{K}$. Therefore, by $\mathcal{A} \in fin_{\propto_2}(\mathbf{K})$, (6) yields $\mathcal{C} = \mathcal{A}$. Hence, $\mathcal{N} \propto_1 \mathcal{A}$. Since $\mathcal{N} \propto_2 \mathcal{B} \propto_1 \mathcal{A}$ holds and $\propto_2$ is antipreserving for $\propto_1$, (19) gives $\mathcal{N} = \mathcal{B}$. Hence, $\mathcal{B} \in fin_{\propto_2}(\mathbf{K})$, that is, $\mathcal{B} \in fin_{\propto_1}(\mathbf{K}) \cap fin_{\propto_2}(\mathbf{K})$, which gives (20). □

None of the assumptions of Lemma 5.8 may be dropped. If $fin_{\propto_i}(\mathbf{K})$ is not $\propto_i$-dense in $\mathbf{K}$ then $fin_{\propto_i+\propto_i}(\mathbf{K})$ is not $\propto_i + \propto_i$-dense in $\mathbf{K}$. If $\propto_1 + \propto_2$ is not a partial order then letting $\propto = \propto_1 + \propto_2^{-1}$ will produce $fin_\propto(\mathbf{K})$ which may not be $\propto$-dense in $\mathbf{K}$. If $\propto_2$ is not antipreserving for $\propto_1$ then Fig. 1 depicts a counterexample to the lemma's thesis.

Figure 1 A counterexample.

The relation $\propto_1$ is represented in Fig. 1 by solid arrows, that is, $\mathcal{M} \propto_1 \mathcal{N}$ holds if and only if there is a solid directed path from $\mathcal{N}$ to $\mathcal{M}$ (this is not a misprint). Similarly, the relation $\propto_2$ is represented by dotted arrows. Put $\mathbf{K} = \{\mathcal{A}_i \mid i \in \omega\} \cup \{\mathcal{B}_i \mid i \in \omega\}$. We have $fin_{\propto_1}(\mathbf{K}) = \{\mathcal{B}_i \mid i \in \omega\}$, and $fin_{\propto_2}(\mathbf{K}) = \{\mathcal{A}_i \mid i \in \omega\}$. One can easily check that all the assumptions of Lemma 5.8 are met, except that $\propto_2$ is preserving for $\propto_1$, that is, $\propto_2$ is not antipreserving for $\propto_1$ ($\propto_1$ *is* antipreserving for $\propto_2$, though). Because $fin_{\propto_1+\propto_2}(\mathbf{K})$ is empty, it cannot be $\propto_1 + \propto_2$-dense in nonempty $\mathbf{K}$.

After all this, the proof of the following fundamental result becomes an easy exercise.

The Completeness Theorem 5.9 Assume that $\mathrm{CPP}(\Gamma_1, \propto_1)$ and $\mathrm{CPP}(\Gamma_2, \propto_2)$ hold, $\propto_1 \circ \propto_2$ is a partial order, $\propto_2$ is antipreserving for $\propto_1$, and Γ_1 and Γ_2 have the intersection property. Then for every $\propto_1$- and $\propto_2$-densly founded $\Sigma \subseteq \forall$ and for every $\varphi \in \forall$,

$$\Sigma \vdash_{\propto_1+\propto_2} \varphi \text{ iff } \varphi \in DA_{\Gamma_1 \cap \Gamma_2}(\Sigma). \tag{21}$$

Proof. Lemma 5.3 yields $\mathrm{CPP}(\Gamma_1 \cap \Gamma_2, \propto_1 + \propto_2)$. Lemma 5.8 gives $\propto_1 + \propto_2$-densly foundedness of every $\propto_1$- and $\propto_2$-densly founded $\Sigma \subseteq \forall$. Application of Theorem 4.2 completes the proof. □

Now, we return to instances of refinements. We will make Theorem 5.9 applicable to the combinations of refinements that were mentioned at the beginning of this section. One can easily check that the following compositions are partial orders:

$$\leq \circ \subseteq,\ \leq \circ \trianglerighteq,\ \trianglerighteq \circ \subseteq,\ \text{ and } (\leq \circ \trianglerighteq) \circ \subseteq .$$

Routine inspection shows that the second refinement of each of these compositions is antipreserving for the first one. We need to show that their CPP's possess the intersection property.

Let Ω be the set of all seven intersections of sets

$$Pos_{\{=\}'},\ Pos_{\{=\}},\ \text{and } QF. \tag{22}$$

Lemma 5.10 Every pair of elements of the set Ω has the intersection property.

Proof. We prove that $Pos_{\{=\}'}$ and QF have the intersection property. (The cases of $Pos_{\{=\}}$ and QF, and $Pos_{\{=\}'}$ and $Pos_{\{=\}}$ have similar proofs; the cases involving their intersections are straightforward).

Let $\Sigma \subseteq QF$ be logically equivalent to a subset of $Pos_{\{=\}'}$. We may assume without loss of generality that all elements of Σ are minimal disjunctions of atomic and negated atomic sentences. It follows from CPP($Pos_{\{=\}'}, \leq$) that $\leq$ preserves the truthfulness of elements of Σ. Therefore, by the minimality of its elements, Σ does not contain occurrences of predicates from $\Re$ in a scope of negation, that is, $\Sigma \subseteq Pos_{\{=\}'}$. Therefore, $\Sigma \subseteq Pos_{\{=\}'} \cap QF$. □

Here are the previously announced $\forall$-complete characterizations of combined entailments by combined default assumption.

Main Theorem 5.11 For every $\Sigma \subseteq \forall$, and every $\varphi \in \forall$,

i. $\Sigma \vdash_{\leq + \subseteq} \varphi$ iff $\varphi \in DA_{Pos_{\{=\}'} \cap QF}(\Sigma)$

ii. $\Sigma \vdash_{\leq + \trianglerighteq} \varphi$ iff $\varphi \in DA_{Pos}(\Sigma)$

iii. $\Sigma \vdash_{\subseteq + \trianglerighteq} \varphi$ iff $\varphi \in DA_{Pos_{\{=\}} \cap QF}(\Sigma)$

iv. $\Sigma \vdash_{\leq + \subseteq + \trianglerighteq} \varphi$ iff $\varphi \in DA_{Pos \cap QF}(\Sigma)$

Proof by direct application of Lemmas 4.3, 3.6.1, 5.10, and Theorem 5.9. □

Note. Because every $\forall$-sentence is logically equivalent to a conjunction of clauses, all instances of Γ used in Theorem 5.11 can be restricted to clauses, that is, QF may be understood as the set of ground clauses, Pos as the set of clauses positive in every symbol in $\Re$ and in equality symbol $=$, and so on.

We end this section with a brief account of some more important consequences of Theorem 5.11.

- the $Pos_{\{=\}'} \cap QF$-default assumption is clause-complete with respect to totally minimal (i.e., predicate-minimal and domain-minimal) semantics. It syntactically characterizes simultaneous application of predicate and domain minimization.

- the Pos-default assumption is clause-complete with respect to predicate-minimal discriminant semantics. For those $\Sigma \subseteq \forall$ which do not contain positive appearances of $=$, the Pos-default assumption is clause-complete with respect to predicate-minimal strongly discriminant semantics (if Σ contains sentences of the form "$a = b$", where a is not identical with b, then Σ does not have strongly discriminant models).

- The $Pos \cap QF$-default assumption is clause-complete with respect to minimal Herbrand (i.e., totally minimal and discriminant) semantics. For those $\Sigma \subseteq \forall$ which do not contain positive occurrences of $=$, the $Pos \cap QF$-default assumption is clause-complete with respect to minimal strongly Herbrand semantics (if Σ contains sentences of the form "$a = b$", where a is not identical with b, then Σ does not have any strongly Herbrand models). Therefore, the $Pos \cap QF$-default assumption may be recognized as the $\forall$-complete version of generalized closed-world assumption *GCWA* (cf. [15] and [23]) and extended closed-world assumption *ECWA* (cf. [4]).

- The $Pos_{\{=\}} \cap QF$-default assumption is clause-complete with respect to Herbrand (i.e., domain-minimal and discriminant) semantics. For those $\Sigma \subseteq \forall$ which do not contain positive occurrences of $=$, the $Pos_{\{=\}} \cap QF$-default assumption is clause-complete with respect to strongly Herbrand semantics.

6 Languages without Equality

What if L does not include the equality symbol? Since the class of strongly Herbrand models, or its subclass, has been widely accepted as the standard semantics for deductive data bases and logic programs, exclusion of $=$ from L was perhaps the simplest way to ensure that a consistent set of clauses has a strongly Herbrand model. In this Section, we use the results of Section 5 to determine how the lack of the equality symbol in L affects the the $\forall$-completeness of the Γ-default assumption.

First, let us note that if $\mathcal{M}$ is a structure for language L with the equality symbol $=$ then $\mathcal{M}$ is also a structure for language L' being a result of removal of $=$ from L. Because the satisfaction relation does not depend on whether the sentence in question is embedded in L or L', it follows that for every set of sentences $\Sigma \subseteq L'$, the class of models of Σ is the same as if Σ was understood as a set of sentences of L. Of course, the set $Cn_{L'}(\Sigma)$ of consequences of Σ in L' is a proper subset of the set $Cn_L(\Sigma)$ of consequences of Σ in L. Nevertheless, a sentence $\varphi \in L'$ is a consequence of Σ within L' if and only if it is a consequence of Σ in L. Therefore, we have the following fact.

Fact 6.1 For every $\Sigma \subseteq L'$,

$$Cn_{L'}(\Sigma) = Cn_L(\Sigma) \cap L'.$$

□

This preservation property of the consequence operation with respect to removing the equality symbol from the underlying language extends over refined entailments. More specifically, for each refinement $\propto$, each $\Sigma \subseteq L'$, and each

$\varphi \in L'$, $\Sigma \vdash_\propto \varphi$ holds independently of whether one embeds Σ and φ in L or in L'. We state this observation in the following form.

Fact 6.2 For every refinement $\propto$ and every $\Sigma \subseteq L'$,

$$\{\varphi \in L' | \Sigma \vdash_\propto \varphi\} = \{\varphi \in L | \Sigma \vdash_\propto \varphi\} \cap L'.$$

□

Unfortunately, the same cannot be said about the Γ-default assumption, for there exists $\Sigma \subseteq \forall \cap L'$ and $\varphi \in \forall \cap L'$ such that $(\Sigma \cup \{\varphi\})_{\Gamma \cap L'} = \Sigma_{\Gamma \cap L'}$, but $(\Sigma \cup \{\varphi\})_\Gamma \neq \Sigma_\Gamma$.

Example 6.3 Let $\Sigma = \{P(c) \vee P(d)\}$, $\varphi = \neg P(c) \vee \neg P(d)$, and $\Gamma = Pos_{\{=\}'}$. We have $(\Sigma \cup \{\varphi\})_{\Gamma \cap L'} = \Sigma_{\Gamma \cap L'}$, but because $\neg(c = d) \in (\Sigma \cup \{\varphi\})_\Gamma \setminus \Sigma_\Gamma$, $(\Sigma \cup \{\varphi\})_\Gamma \neq \Sigma_\Gamma$. □

Therefore, in general, one cannot restrict Γ to $\Gamma \cap L'$ while switching from L to L' without putting the $\forall$-completeness of the Γ-default assumption at risk. For some Γ's it is possible, though. To show this we need the following technical lemmas.

Lemma 6.4 For each $\Sigma \subseteq \forall \cap L'$ and $\varphi \in Pos_{\{=\}}$,

$$\Sigma \vdash_{\underset{\sim}{\downarrow}} \varphi \text{ iff } \Sigma \vdash \varphi.$$

Proof. $\Sigma \vdash_{\underset{\sim}{\downarrow}} \varphi$ iff [by Theorem 4.4.iv] $(\Sigma \cup \{\varphi\})_{Pos_{\{=\}}} = \Sigma_{Pos_{\{=\}}}$ iff [since $\varphi \in (\Sigma \cup \{\varphi\})_{Pos_{\{=\}}}$] $\varphi \in \Sigma_{Pos_{\{=\}}}$ iff $\Sigma \vdash \varphi$. □

Definition 6.5 Operation $'$ on $\forall$-formulas of L and sets of $\forall$-formulas of L is defined by induction as follows.

i. for every term t of L,
$$(t = t)' = true$$

ii. for every pair t, s of non-identical terms of L,
$$(t = s)' = false$$

iii. for every $\forall$-formula φ of $\forall$ without occurrences of $=$,
$$\varphi' = \varphi$$

iv. for all quantifier-free formulas φ, ψ of L,
$$(\neg\varphi)' = \neg\varphi'$$
$$(\varphi \wedge \psi)' = \varphi' \wedge \psi'$$

v. for every quantifier-free formula φ of L,

$$(\forall\vec{x}\varphi)' = \forall\vec{x}\varphi'$$

vi. for every $\Gamma \subseteq L$,

$$\Gamma' = \{\varphi' \mid \varphi \in \Gamma\}.$$

□

Lemma 6.6 For each strongly discriminant structure $\mathcal{M}$ for L, and each $\varphi \in QF$,

$$\mathcal{M} \models \varphi \text{ iff } \mathcal{M} \models \varphi'.$$

Proof by routine induction, using Definition 6.5, i - iv. □

Lemma 6.7 For every $\Sigma \subseteq \forall \cap L'$ and $\varphi \in QF$,

$$\Sigma \vdash_{\underline{\geq}} \varphi \text{ iff } \Sigma \vdash_{\underline{\geq}} \varphi'.$$

Proof. $\Sigma \vdash_{\underline{\geq}} \varphi$ iff [because Σ does not contain occurrences of $=$] φ is true in all strongly discriminant models of Σ iff [by Lemma 6.6] φ' is true in all strongly discriminant models of Σ iff $\Sigma \vdash_{\underline{\geq}} \varphi'$. □

Lemma 6.8 Assume that $\Gamma \subseteq \Gamma' \cup Pos_{\{=\}}$. For every Σ, $\Pi \subseteq \forall \cap L'$,

$$\Sigma_\Gamma = \Pi_\Gamma \text{ iff } \Sigma_{\Gamma'} = \Pi_{\Gamma'}. \tag{23}$$

Proof. Assume $\varphi \in \Gamma$. We are going to show that $\Sigma \vdash \varphi$ iff $\Sigma \vdash \varphi'$, and $\Pi \vdash \varphi$ iff $\Pi \vdash \varphi'$. This will prove (23).

Case 1: $\varphi \in \Gamma'$, is obvious since [by Definition 6.5.vi] $\varphi = \varphi'$ in this case.

Case 2: $\varphi \in Pos_{\{=\}} \cap QF$. $\Sigma \vdash \varphi$ iff [by Lemma 6.4] $\Sigma \vdash_{\underline{\geq}} \varphi$ iff [by Lemma 6.7] $\Sigma \vdash_{\underline{\geq}} \varphi'$ iff [by Lemma 6.4; note that by Definition 6.5, $\varphi' \in Pos_{\{=\}}$] $\Sigma \vdash \varphi'$. Similarly, $\Pi \vdash \varphi$ iff $\Pi \vdash \varphi'$.

Case 3: $\varphi \in Pos_{\{=\}} \setminus QF$. Let $\varphi = \forall\vec{x}\psi(\vec{x})$, where $\psi(\vec{x})$ is quantifier-free. $\Sigma \vdash \forall\vec{x}\psi(\vec{x})$ iff $\Sigma \vdash \psi(\vec{c})$, where $\vec{c}$ is a sequence of constants not in Σ with $\vec{x}$ and $\vec{c}$ having the same pattern of repetitions, iff [using the same argument as in Case 2] $\Sigma \vdash \psi(\vec{c})'$ iff $\Sigma \vdash \forall\vec{x}\psi(\vec{x})'$ iff [by Definition 6.5.v] $\Sigma \vdash (\forall\vec{x}\psi(\vec{x}))'$. Similarly, $\Pi \vdash \varphi$ iff $\Pi \vdash \varphi'$. □.

Now, we are ready to prove the $\forall$-completeness of the following instances the Γ-default assumption with respect to instances of entailment $\vdash_\alpha$.

Theorem 6.9 For every $\Sigma \subseteq \forall \cap L'$ and every $\varphi \in \forall \cap L'$,

i. $\Sigma \vdash_{\leq+\underline{\geq}} \varphi$ iff $\varphi \in DA_{Pos\cap L'}(\Sigma)$

ii. $\Sigma \vdash_{\subseteq+\underline{\geq}} \varphi$ iff $\varphi \in DA_{QF\cap L'}(\Sigma)$

iii. $\Sigma \vdash_{\leq + \subseteq + \trianglerighteq} \varphi$ iff $\varphi \in DA_{Pos \cap QF \cap L'}(\Sigma)$

Proof. Let us note that the assumption $\Gamma \subseteq \Gamma' \cup Pos_{\{=\}}$ is met for these instances of Γ: Pos, $QF \cap Pos_{\{=\}}$, and $Pos \cap QF$. Respective instances of Γ' are: $Pos \cap L'$, $QF \cap L'$, and $Pos \cap QF \cap L'$. The rest of proof is a direct application of Lemma 6.8 to theorem 5.11. □

$DA_{Pos \cap L'}$ was first introduced in [19] as an instance of cwa_S, and proven complete with respect to entailment $\vdash_{\leq + \trianglerighteq}$ in [20].

As noted at the end of Section 5, QF in Theorem 6.9 may be replaced with the set of ground clauses, Pos with the set of clauses positive in every symbol in $\Re$, and $Pos \cap QF$ with the set of ground clauses positive in every symbol in $\Re$.

We conclude this Section with the application of Theorem 6.9 to the evaluation of the degree of completeness of published in [23] extended version $GCWA^*$ of Minker's $GCWA$.

Let us note that the condition involved in the definition of Γ-default assumption, namely

$$(\Sigma \cup \{\varphi\})_\Gamma = \Sigma_\Gamma \tag{24}$$

is equivalent to

$$\bigwedge_{\gamma \in \Gamma} \Sigma \vdash \neg\varphi \vee \gamma \text{ implies } \Sigma \vdash \gamma, \tag{25}$$

which constitutes the familiar scheme whose instances occur in definitions of Minker style closed-world assumptions. For $\Gamma = Pos \cap QF \cap L'$ (the set of ground positive clauses without occurrences of $=$), and φ restricted to $QF \cap L'$, (25) expresses the $GCWA^*(\Sigma)$, that is, by (10),

$$\bigwedge_{\Sigma \subseteq \forall} GCWA^*(\Sigma) = DA_{Pos \cap QF \cap L'}(\Sigma) \cap QF \cap L'. \tag{26}$$

Therefore, by Theorem 6.9.iii, $GCWA^*$ is ground-complete with respect to minimal strongly Herbrand entailment (cf. [23] for an early proof of this fact).

One should note, however, that although for some refinements $\propto$ that do not involve $\subseteq$ (and consequently, respective $\propto$-finest semantics are not restricted to domain-minimal semantics), their complete preservation properties Γ satisfy

$$\bigwedge_{\forall x \varphi(x) \in \forall} \forall x \varphi(x) \in DA_\Gamma(\Sigma) \text{ iff } \varphi(c) \in DA_\Gamma(\Sigma),$$

(for example, take $\propto = \leq$, and $\Gamma = Pos_{\{=\}'}$), where c is a new constant symbol from beyond L, the same is not true for $\leq + \subseteq + \trianglerighteq$. Therefore, the

ground completeness (26) of $GCWA^*$ does not imply its $\forall$-completeness. Indeed, $GCWA^*$ is not $\forall$-complete with respect to minimal strongly Herbrand semantics, except of course, for those Σ's which admit elimination of universal quantifiers. Here is a counterexample.

Example 6.10 Put $\Sigma = \{P(a) \vee Q(b)\}$. One can check that $GCWA^*(\Sigma) \vdash \neg P(c)$. On the other hand, $\forall x \neg P(x)$ is inconsistent with Σ. Hence, $\varphi(c)$ cannot be used as a valid replacement for $(\forall x)(\varphi(x))$ when $GCWA^*$ is used. Consequently, $GCWA^*(\Sigma) \not\vdash (\forall x)(\neg P(x) \vee \neg Q(x))$, even though $\Sigma \vdash_{\leq + \subseteq + \trianglerighteq} (\forall x)(\neg P(x) \vee \neg Q(x))$. Of course, by Theorem 6.9.iii, $(\forall x)(\neg P(x) \vee \neg Q(x)) \in DA_{Pos \cap QF \cap L'}(\Sigma)$. □

This fact makes $GCWA^*$ strictly weaker than $DA_{Pos \cap QF \cap L'}$. Similar argument shows that also $ECWA$ of [4] is strictly weaker than $DA_{Pos \cap QF \cap L'}$; it turns out that the published in [4] proof of $\forall$-completeness of $ECWA$ works only for cases of Σ which admit elimination of quantifiers.

7 Few Words about the Roots

Default rules of inference were introduced by Reiter in [17], also the inventor of the closed-world assumption ([16]), and quickly became a very fashionable subject of research. At the same time, McCarthy in his famous paper [14] about circumscription initiated investigations of first-order minimal models. Second-order variant of circumscription was proven complete with respect to minimal-model semantics by Lifschitz in [9]. A restricted form of scheme (25) was first used by Minker in [15], with subsequent ground-complete extensions by Yahya and Henschen ([23]), and Gelfond, Przymusińska and Przymusiński ([4]). Credit for the first proof of dense foundedness is due to Bossu and Siegel ([2]).

Marczewski in [13] and Lyndon in [10] (followed by [12]) were first to study preservations of positive sentences. The consequence operation defined by

$$(\Sigma \cup \{\varphi\})_\forall = \Sigma_\forall \tag{27}$$

was first used by Kaiser in [7] and later by Henrard, who proved in [5] that certain iterative use of (27) completely characterizes the generic-model semantics of Robinson's model-theoretic forcing of [18]. A generalization (24) of scheme (27) was first applied to analysis of the closed-world assumption in [19], with its $\forall$-completeness proven in [20].

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