P – a program with input I that runs in time $t(I)$

D – the set of all valid inputs for P (I)

size (I) – size of input I

$size : D \rightarrow \mathbb{N}$

D_n – the set of all valid inputs of size n

$$D_n = \{I \in D \mid size(I) = n\}$$

Running time as a function of size of input

$$Time : \mathbb{N} \rightarrow \mathbb{R}^+$$

Worst – case running time

$$T(n) = \max\{t(I) \mid I \in D_n\}$$

Average – case running time

Given probability distribution p_n on D_n,

$$T_{avg}(n) = \sum\{t(I) \times p_n(I) \mid I \in D_n\}$$

In the case of uniform distribution of probability :

$$T_{avg}(n) = \frac{\sum\{t(I) \mid I \in D_n\}}{|D_n|}$$

Example.

QuickSort :

D – the set of all permutations of some initial interval of the set of natural numbers $\mathbb{N}$

size (I) = number of elements in I (to be sorted)

D_n – the set of all permutations of {0, ..., n – 1}

$$|D_n| = n!$$

Worst – case running time :

$$T(n) \sim n^2$$

The worst input of size n is the sequence
< 0, ..., n – 1 > .

Average – case running time :

Assume that all inputs of size
n to QuickSort are equally likely $\left(\text{with probability } \frac{1}{n!}\right)$.

$$T_{avg}(n) \sim n \log_2 n$$

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Plot[Tooltip[{x Log[x], x^2}], {x, 0, 10}]
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