CSC 311
Data Structures
Fall ’14

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CSUDH

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$T(n)$, max input's size $Max(t)$, and its derivative.

Let $T(n)$ be a running time of some program P. Let us assert that $T(n)$ is a growing function.

Definition of $Max(t)$

$Max(t)$ is defined as the maximum size n of input for which $T(n) \leq t$.

The inverse of a running time

Under the above assertion, $Max(t)$ is the inverse of $T(n)$, that is,

$$t = T(n) \text{ iff } n = Max(t). \tag{1}$$

In particular, $Max(t)$ is a growing function as well.

$T(n)$, max input's size $Max(t)$, and its derivative.

The following fact holds for every differentiable growing function f:

$$(f_{inverse})'(x) = \frac{1}{f'(f_{inverse}(x))}, \tag{2}$$

where $f_{inverse}$ is the inverse of f (it exists since f is a growing function) and f' is the derivative of f.

In particular,

$$Max'(t) = \frac{1}{T'(Max(t))}. \tag{3}$$

$T(n)$, max input's size $Max(t)$, and its derivative.

The derivative $Max'(t)$ of $Max(t)$ seems like a good measure of return on investment of a faster computer (or - equivalently - longer wait) for program P.

- It tells how fast (or slow) the maximum size of tractable input to P will grow with the increase of the computer's speed.
- So, the larger $Max'(t)$ the more cost effective it is at point t to run P on a faster computer.
- And vice versa: the smaller $Max'(t)$ the more wasteful it is at point t to run P on a faster computer.

$T(n)$, max input's size $Max(t)$, and its derivative.

When the measure $Max'(t)$ is decreasing then it might be insightful to consider also the reciprocal $\frac{1}{Max'(t)}$ of $Max'(t)$.

- The measure $\frac{1}{Max'(t)}$ tells how much faster (or longer) the program P must be executed in order to accomplish the unit increase of tractable input to P.
- So, the larger the $\frac{1}{Max'(t)}$ the more costly it is at point t to run P on even a slightly larger input.

$T(n)$, max input's size $Max(t)$, and its derivative.

By (3),

$$\frac{1}{Max'(t)} = T'(Max(t)). \tag{4}$$

$T(n)$, max input's size $Max(t)$, and its derivative.

Finding the Θ characterization of $f'(t)$ and of $\frac{1}{f'(t)}$ from the Θ characterization of $f(t)$ requires some extra assumption that f and its Θ benchmark (representative) satisfy assumptions of the de l'Hôpital rule.

Theorem

Let f and g be positive, increasing, differentiable functions that both converge to 0 or both diverge to ∞ as their arguments diverge to ∞. Assume that $\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ exists. Then

$$f \in \Theta(g) \equiv f' \in \Theta(g') \equiv \frac{1}{f'} \in \Theta(\frac{1}{g'}).$$

$T(n)$, max input's size $Max(t)$, and its derivative.

Below, several examples of $T(n)$ and corresponding $Max(t)$ and the derivative $Max'(t)$ are described. For the cases viii through xiii the reciprocals $\frac{1}{Max'(t)}$ are included. All cases i through xiii may be considered benchmark cases.

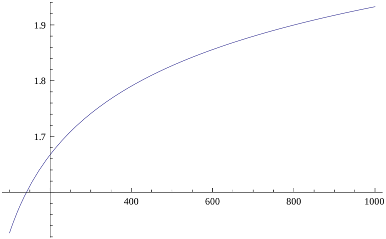
Particularly important are cases: i, iii, vii, viii, ix, xi, and xii.

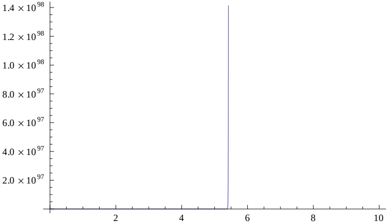
Note different scales used in graphs of sample functions below.

$T(n)$, max input's size $Max(t)$, and its derivative.

$$T(n) \in \Theta(\log \log n) \ldots Max(t) \in \Theta(a^{b^t}); \text{ for some } a, b > 1$$

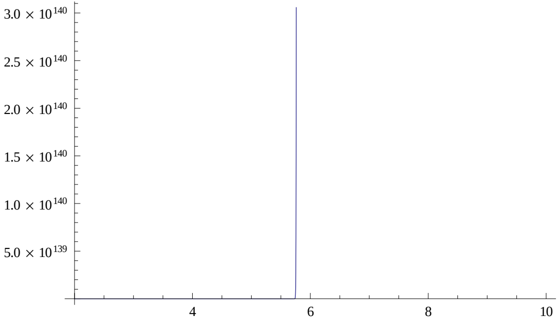
Here are graphs of $\ln \ln n$ and e^{e^t}:





$$Max'(t) \in \Theta(a^{b^t} b^t)$$

Here is a graph of $e^{e^t} e^t = e^{e^t + t}$:

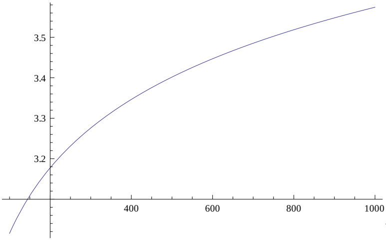


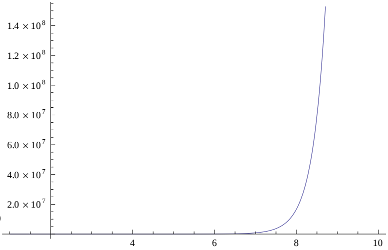
In this case, the larger t the (dramatically) more it pays off to run P on a faster computer.

ii.
$$T(n) \in \Theta(\frac{\log n}{\log \log n}) \ldots Max(t) \in \Omega(at)^{at}) \cap O((bt)^{bt});$$

for some $a, b > 1$

Here are graphs of $\frac{\log n}{\log \log n}$ and t^t:

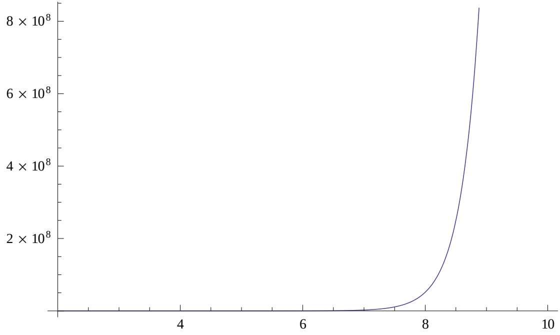




$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Omega((at)^{at} \ln t) \cap O((bt)^{bt} \ln t)$$

Here is a graph of $t^t \ln t$:



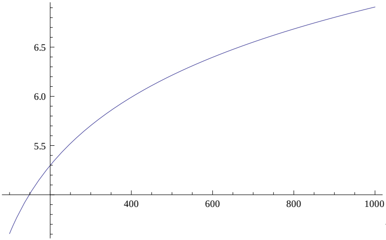
In this case, the larger t the (significantly) more it pays off to run P on a faster computer.

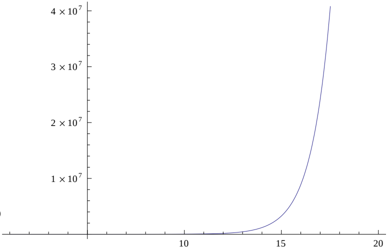
$T(n)$, max input's size $Max(t)$, and its derivative.

iii

$$T(n) \in \Theta(\log n) \ldots Max(t) \in \Theta(a^t); \text{ for some } a > 1$$

Here are graphs of $\ln n$ and e^t:





$$Max'(t) \in \Theta(a^t)$$

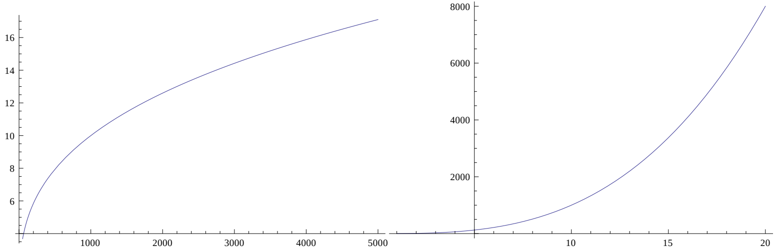
See above for a graph of e^t.

In this case, the larger t the (significantly) more it pays off to run P on a faster computer.

iv

$$T(n) \in \Theta(\sqrt[3]{n}) \ldots Max(t) \in \Theta(t^3)$$

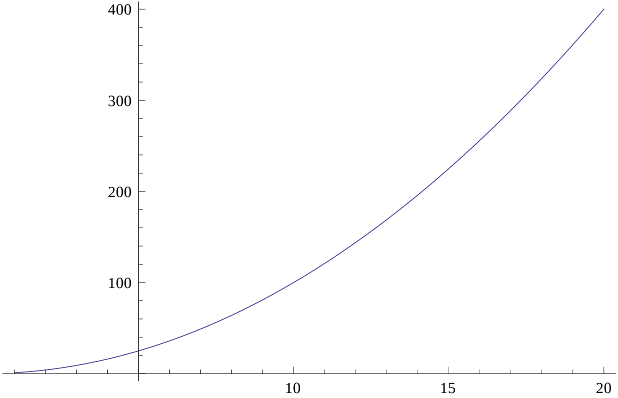
Here are graphs of $\sqrt[3]{n}$ and t^3:



$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Theta(t^2)$$

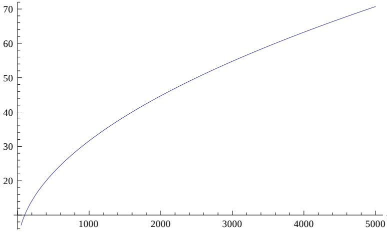
Here is a graph of t^2:

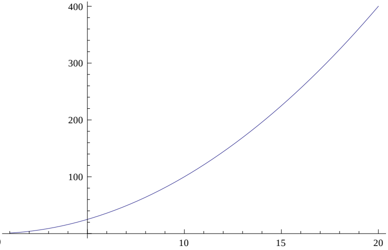


In this case, the larger t the more it pays off to run `P` on a faster computer.

$$T(n) \in \Theta(\sqrt{n}) \ldots Max(t) \in \Theta(t^2)$$

Here are graphs of $\sqrt{n}$ and t^2:

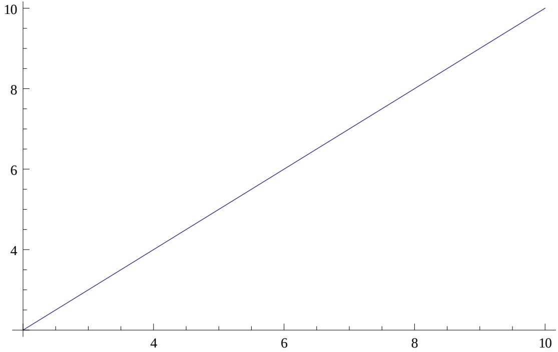




$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Theta(t)$$

Here is a graph of t:

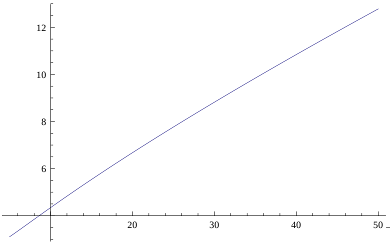


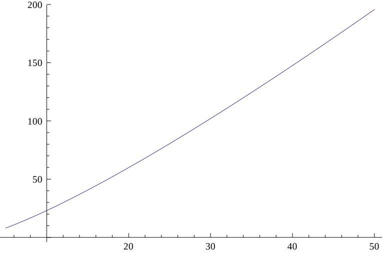
In this case, the larger t the more it pays off to run `P` on a faster computer.

vi

$$T(n) \in \Theta(\frac{n}{\log n}) \ldots Max(t) \in \Theta(t \log t)$$

Here are graphs of of $\frac{n}{\log n}$ and $t \ln t$:

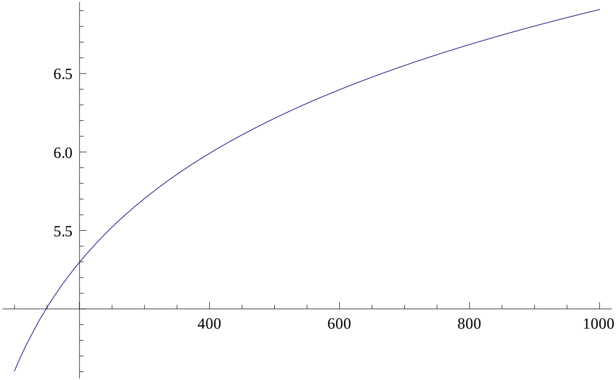




$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Theta(\log t)$$

Here is a graph of $\ln n$:



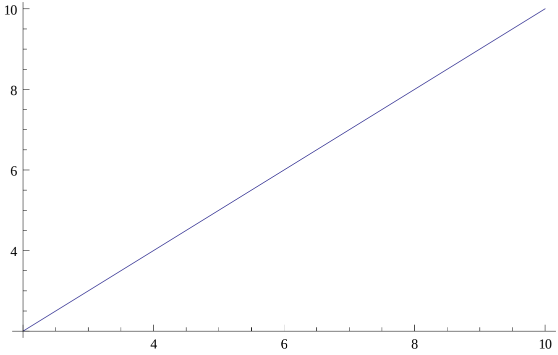
In this case, the larger t the (moderately) more it pays off to run P on a faster computer.

$T(n)$, max input's size $Max(t)$, and its derivative.

vii

$$T(n) \in \Theta(n) \ldots Max(t) \in \Theta(t)$$

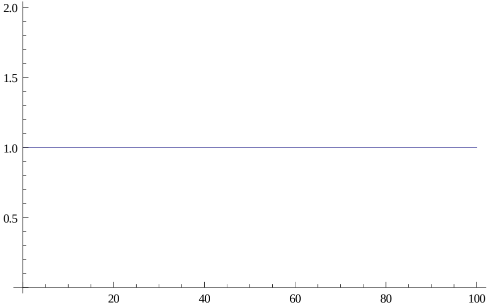
Here is a graphs of n and t:



$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Theta(1)$$

Here is a graph of 1:



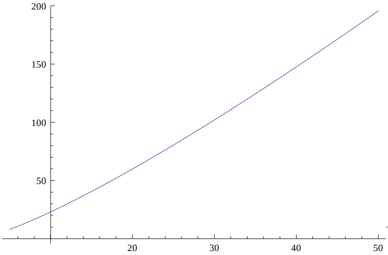
In this case, the increase of the maximum size of input in function of speed of the computer is constant for all t, so the payoff for running P on a faster computer remains roughly the same for all sizes of its inputs.

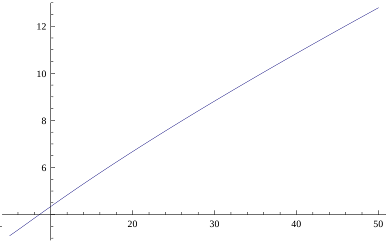
$T(n)$, max input's size $Max(t)$, and its derivative.

viii

$$T(n) \in \Theta(n \log n) \ldots Max(t) \in \Theta(\frac{t}{\log t})$$

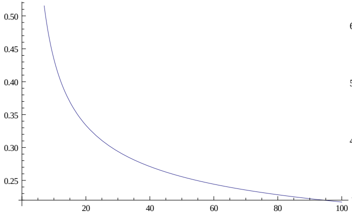
Here are graphs of $n \log n$ and $\frac{t}{\log t}$:

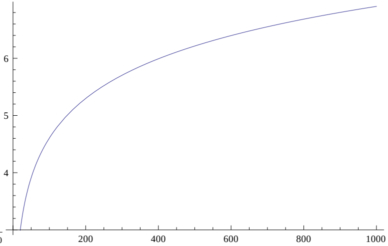




$$Max'(t) \in \Theta(\frac{1}{\log t}); \quad \frac{1}{Max'(t)} \in \Theta(\log t)$$

Here are graphs of $\frac{1}{\log t}$ and $\log t$:



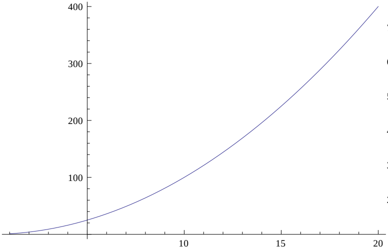


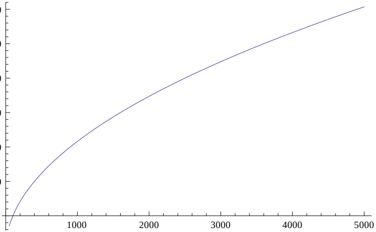
In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (slightly) more does it cost to accomplish the unit increase of the tractable input to P.

ix

$$T(n) \in \Theta(n^2) \ldots Max(t) \in \Theta(\sqrt{t})$$

Here are graphs of n^2 and $\sqrt{t}$:

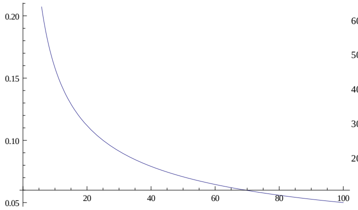


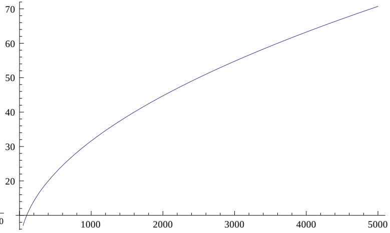


$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Theta(\frac{1}{\sqrt{t}}); \ \frac{1}{Max'(t)} \in \Theta(\sqrt{t})$$

Here are graphs of $\frac{1}{\sqrt{t}}$ and $\sqrt{t}$:



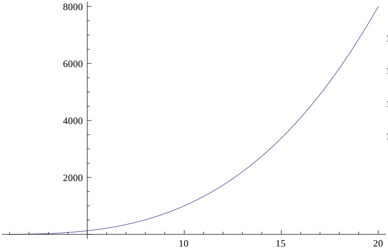


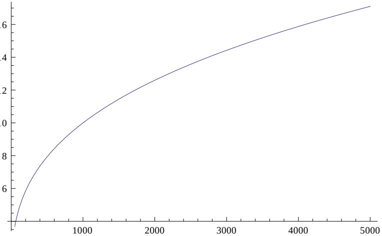
In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (moderately) more does it cost to accomplish the unit increase of the tractable input to P.

$T(n)$, max input's size $Max(t)$, and its derivative.

- $$T(n) \in \Theta(n^3) \ldots Max(t) \in \Theta(\sqrt[3]{t})$$

  Here are graphs of n^3 and $\sqrt[3]{t}$:

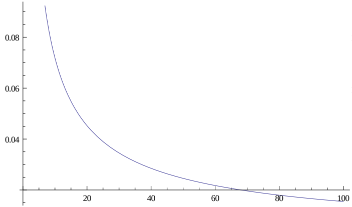


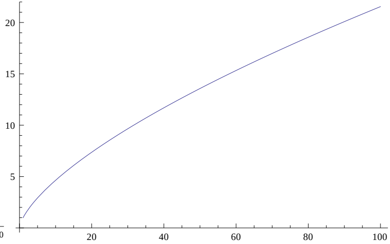


$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Theta\left(\frac{1}{\sqrt[3]{t^2}}\right); \quad \frac{1}{Max'(t)} \in \Theta(\sqrt[3]{t^2})$$

Here are graphs of $\frac{1}{\sqrt[3]{t^2}}$ and $\sqrt[3]{t^2}$:





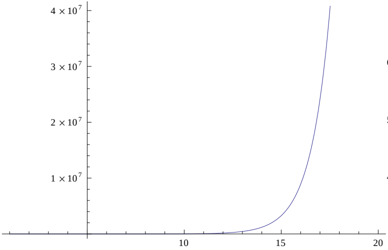
In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (moderately) more does it cost to accomplish the unit increase of the tractable input to P.

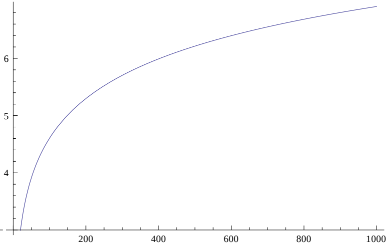
$T(n)$, max input's size $Max(t)$, and its derivative.

xi

$$T(n) \in \Theta(a^n) \ldots Max(t) \in \Theta(\log t); \text{ for all } a > 1$$

Here are graphs of e^n and $\ln t$:

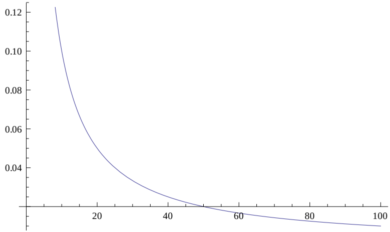


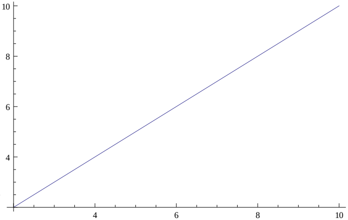


$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Theta(\frac{1}{t}); \frac{1}{Max'(t)} \in \Theta(t)$$

Here are graphs of $\frac{1}{t}$ and t:





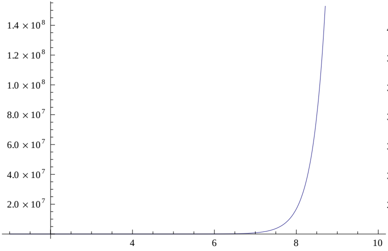
In this case, the larger t the less it pays off to run P on a faster computer. More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P.

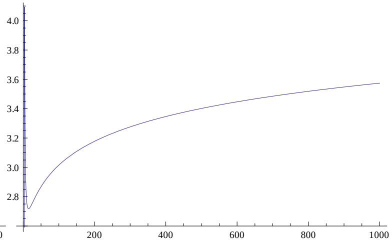
$T(n)$, max input's size $Max(t)$, and its derivative.

xii

$$T(n) \in \Theta((an)^{bn}) \ldots Max(t) \in \Theta(\frac{\log t}{\log \log t}); \text{ for all } a, b > 1$$

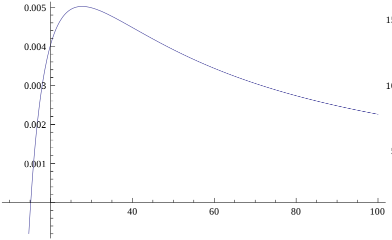
Here are graphs of n^n and $\frac{\log t}{\log \log t}$:

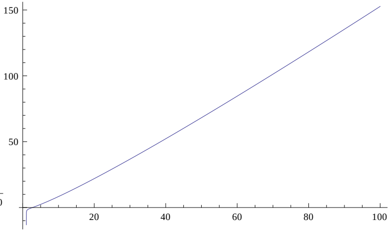




$$Max'(t) \in \Theta(\frac{1}{t \ln \ln t} - \frac{1}{t(\ln \ln t)^2}) = \Theta(\frac{1}{t \ln \ln t}); \ \frac{1}{Max'(t)} \in \Theta(t \ln l$$

Here are graphs of $\frac{1}{t \ln \ln t} - \frac{1}{t(\ln \ln t)^2}$ and $t \ln \ln t$:





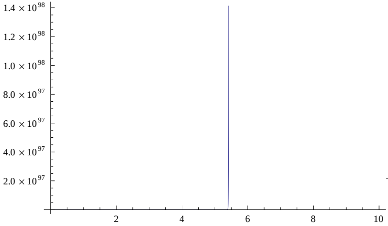
In this case, the larger t the (dramatically) less it pays off to run P on a faster computer. More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P.

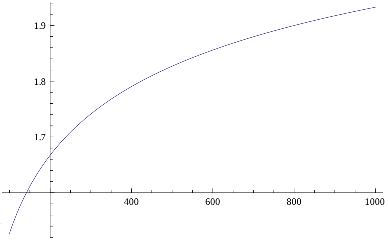
$T(n)$, max input's size $Max(t)$, and its derivative.

xiii

$$T(n) \in \Theta(a^{b^n}) \ldots Max(t) \in \Theta(\log\log t); \text{ for all } a, b > 1$$

Here are graphs of e^{e^n} and $\log\log t$:

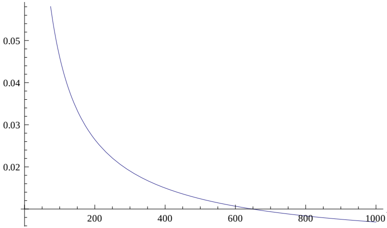


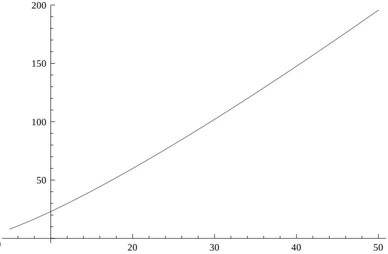


$T(n)$, max input's size $Max(t)$, and its derivative.

$$Max'(t) \in \Theta(\frac{1}{t \ln t}); \ \frac{1}{Max'(t)} \in \Theta(t \ln t)$$

Here are graphs of $\frac{1}{t \ln t}$ and $t \ln t$:





In this case, the larger t the (dramatically) less it pays off to run P on a faster computer.More insightfully, the larger the t the (significantly) more does it cost to accomplish the unit increase of the tractable input to P.

END