# Solving a recurrence relation for the average – case running time A[n] for QuickSort:

$$A[n] = n - 1 + \frac{1}{n} \sum_{i=0}^{n-1} (A[i] + A[n-1-i])$$

$$A[0] = 0$$

$$A[1] = 0$$

We will calculate a close approximation

$$A[n] \approx$$

$$1.3862943611198906` \,(n+1)\, \text{Log2}[n] - 2.8455686701969345` \, n + 2.1544313298030655` + \frac{1}{n}\, 0.8333333333333334` - \frac{0.16666666666666666`}{n^2}$$

the difference between which and the exact A[n] converges to 0 faster than $\frac{1}{n^2}$.

The part highlighted yellow and its derivation is mandatory for all students; the rest of the aproximation and its derivation is optional – they are shown here for illustration purposes only.

(The textbook approximation was

$$A[n] \approx 1.386\, n\, \text{Log2}[n] - 2.846\, n$$

– good enough if you don't want to be too accurate, although

$$A[n] \approx 1.386\, (n+1)\, \text{Log2}[n] - 2.846\, n$$

is somewhat better.)

We have:

$$\sum_{i=0}^{n-1} (A[i] + A[n-1-i]) =$$

$$= \sum_{i=0}^{n-1} A[i] + \sum_{i=0}^{n-1} A[n-1-i] =$$

[putting $j = n - 1 - i$; $j \in \{0, \ldots, n-1\}$]

$$= \sum_{i=0}^{n-1} A[i] + \sum_{j=0}^{n-1} A[j] =$$

$$= \sum_{i=0}^{n-1} A[i] + \sum_{i=0}^{n-1} A[i] =$$

$$= 2 \sum_{i=0}^{n-1} A[i]$$

So,

$$A[n] = n - 1 + \frac{2}{n} \sum_{i=0}^{n-1} A[i]$$

$$A[0] = 0$$

$$A[1] = 0$$

Notation change to use Mathematica

$$A_n = n - 1 + \frac{2}{n} \sum_{i=0}^{n-1} A_i$$

$$A_{n-1} = n - 2 + \frac{2}{n-1} \sum_{i=0}^{n-2} A_i$$

$$n \times A_n - (n-1) A_{n-1}$$

$$-(-1+n)\left(-2+n+\frac{2\sum_{i=0}^{-2+n} A_i}{-1+n}\right) + n\left(-1+n+\frac{2\sum_{i=0}^{-1+n} A_i}{n}\right)$$

$$\text{Expand}\left[-(-1+n)\left(-2+n+\frac{2\sum_{i=0}^{-2+n} A_i}{-1+n}\right) + n\left(-1+n+\frac{2\sum_{i=0}^{-1+n} A_i}{n}\right)\right]$$

$$-2+2n+\frac{2\sum_{i=0}^{-2+n} A_i}{-1+n} - \frac{2n\sum_{i=0}^{-2+n} A_i}{-1+n} + 2\sum_{i=0}^{-1+n} A_i$$

$$\text{Factor}\left[-2+2n+\frac{2\sum_{i=0}^{-2+n} A_i}{-1+n} - \frac{2n\sum_{i=0}^{-2+n} A_i}{-1+n} + 2\sum_{i=0}^{-1+n} A_i\right]$$

$$2\left(-1+n-\sum_{i=0}^{-2+n} A_i + \sum_{i=0}^{-1+n} A_i\right)$$

$$2\left(-1+n-\sum_{i=0}^{-2+n} A_i + \sum_{i=0}^{-1+n} A_i\right) = 2\left(-1+n-\sum_{i=0}^{-2+n} A_i + \sum_{i=0}^{-2+n} A_i + \sum_{i=-1+n}^{-1+n} A_i\right)$$

$$2\left(-1+n-\sum_{i=0}^{-2+n} A_i+\sum_{i=0}^{-2+n} A_i+\sum_{i=-1+n}^{-1+n} A_i\right)$$

$2\,(-1+n+A_{-1+n})$

**$2\,(-1+n+A_{-1+n})$**

**So,**

**$n \times A_n - (n-1)\,A_{n-1} = 2\,(-1+n+A_{-1+n})$**

**or**

**$n \times A_n = (n+1)\,A_{n-1} + 2\,(-1+n)$**

**RSolve$\left[\left\{\frac{A[n]}{n+1} == \frac{A[n-1]}{n} + \frac{2\,(n-1)}{n\,(n+1)},\ A[0] == 0,\ A[1] == 0\right\},\ A[n],\ n\right]$**

{{A[n] → 2 (EulerGamma − 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])}}

So, **the exact solution** of the original recurrence relation on the average – case running time of QuickSort is :

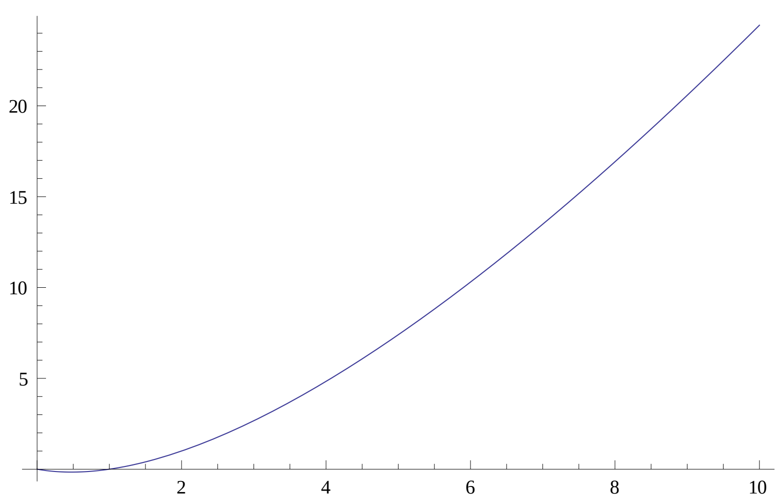
A[n] =

2 (EulerGamma − 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])

**We will approximate it in more humane terms.**

**Here is the graph of the exact solution :**

**Plot[{2 (EulerGamma − 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])}, {n, 0, 10}]**



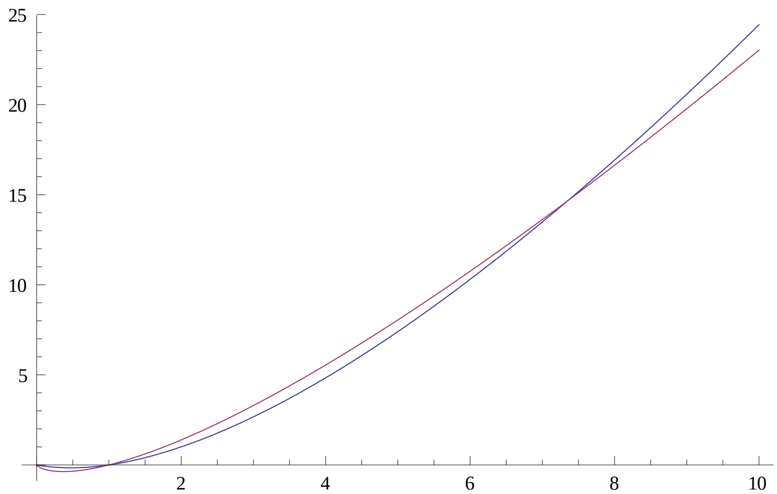
**The above shape strongly suggests that it ' s growth rate is about**

**n Log[n].**

**Here is the plot of both functions :**

```
In[15]:= Plot[
 Tooltip[{2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]),
   n Log[n]}], {n, 0, 10}]
```

Out[15]=




**Indeed, the ratio of the two converges :**

```
In[18]:= Limit[{(2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])) /
   (n Log[n])}, n → ∞]
```

Out[18]= {2}

**In other words, using Sedgewick ' s notation ~ :**

**A[n] ~ 2 n Log[n]**

**So,**

$A[n] \in \Theta(n \, Log2[n])$

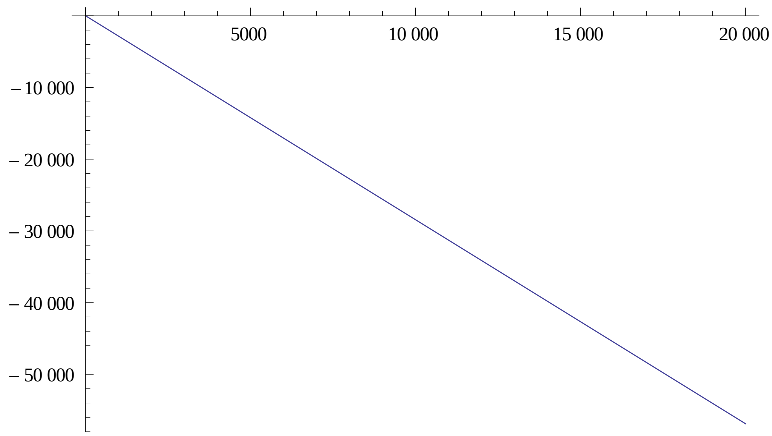
**Now, let ' s find some other slower growing terms of exact solution A[n].**

**When we compare the exact solution A[n] to its Θ characterization 2 n Log[n], we can notice that the difference between the two approximates a linear function in variable n.**

**Here is a graph of that difference :**

In[19]:=
```
Plot[{(2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])) -
    (2 n Log[n])}, {n, .001, 20000}]
```

Out[19]=




**In order to find a coefficient of that (approximately) linear function (the slope of the above line), we calculate limit of the mentioned above difference divided by n, that is :**

In[20]:=
```
Limit[{(2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]) -
      2 n Log[n]) / n}, n → ∞]
```

Out[20]= {2 (-2 + EulerGamma)}

In[4]:= **N[2 (-2 + EulerGamma)]**

Out[4]= -2.84557

**-2.8455686701969345`**

**N[Log[4]]**

1.38629

**1.3862943611198906`**

**This yields the textbook ' s approximation : .**

**2 n Log[n] + 2 (-2 + EulerGamma) n = 1.3862943611198906` n Log2[n] - 2.8455686701969345` n**

**But the function**

**2 n Log[n] + 2 (-2 + EulerGamma) n**

**does not approximate A[n] well enough when n → ∞ because the difference between the two diverges to ∞ :**

In[21]:=
```
Limit[{2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]) -
    2 n Log[n] - 2 (-2 + EulerGamma) n}, n → ∞]
```

Out[21]= {∞}

So we need to inject another adjustment to our approximation. We will inject a term of the form constant × Log2[n], because the above mentioned difference is in Θ (Log2[n]), as the following limit proves :

In[22]:=

```
Limit[{(2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]) -
    Log[4] n Log2[n] - 2 (-2 + EulerGamma) n) / Log[n]}, n → ∞]
```

{2}

This gives us this improved approximation of A[n] :

2 n Log[n] + 2 (-2 + EulerGamma) n + 2 Log[n]

or, after grouping the first an the last term,

2 (n + 1) Log[n] + 2 (-2 + EulerGamma) n

It yields the approximation

1.386 (n + 1) Log2[n] - 2.846 n

mentioned at the begining of this file.

This one is close, but still need another adjustment, constant at this time. We find that constant by letting Mathematica compute the limit of the difference between the exact A[n] and our approximation :

In[23]:=

```
Limit[{2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]) -
    2 (n + 1) Log[n] - 2 (-2 + EulerGamma) n}, n → ∞]
```

Out[23]= {1 + 2 EulerGamma}

= 1 + 2 EulerGamma

N[1 + 2 EulerGamma]

2.15443

So, the function

Log[4] (n + 1) Log2[n] + 2 (-2 + EulerGamma) n + (1 + 2 EulerGamma) =
= 1.3862943611198906` (n + 1) Log2[n] - 2.8455686701969345` n + 2.1544313298030655`
asympthotically approaches A[n] as n → ∞. In particular, the difference between the two converges to 0, as the following limit shows :

In[24]:=

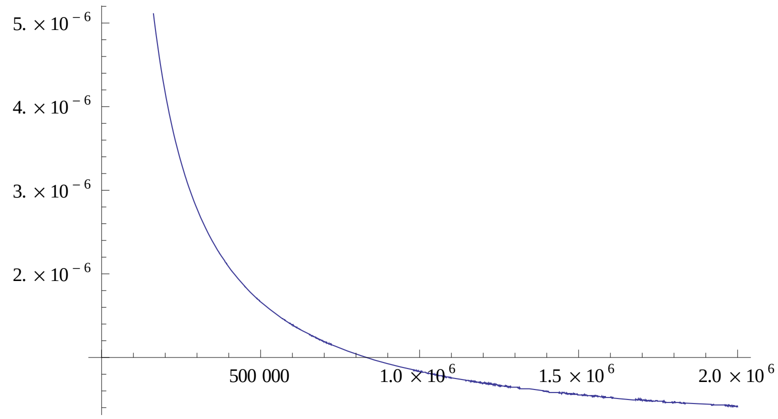
```
Limit[{2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]) -
    (2 n Log[n] + 2 (-2 + EulerGamma) n + 2 Log[n] + (1 + 2 EulerGamma))}, n → ∞]
```

Out[24]= {0}

**Here is a graph of the difference between the exact A[n] and our approximation:**

```
In[26]:= Plot[{2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]) -
    (2 n Log[n] + 2 (-2 + EulerGamma) n + 2 Log[n] + (1 + 2 EulerGamma))}, {n, 2, 2000000}]
```

Out[26]=



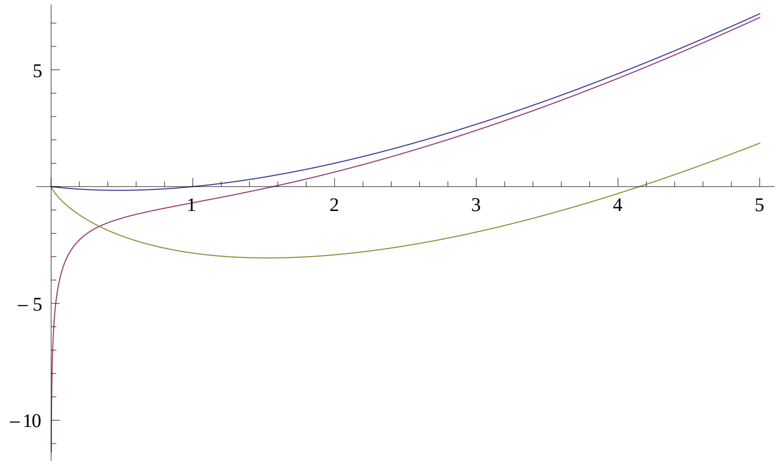

**So, it is close, indeed.**

**Here is our approximation, again, in even more humane terms:**

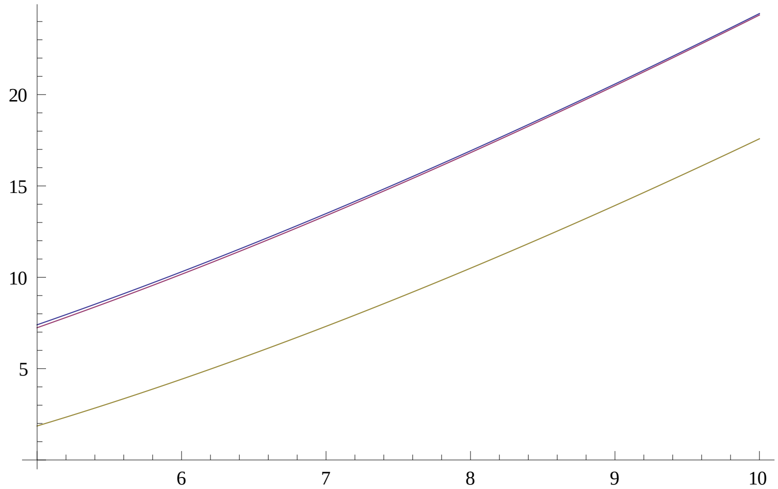
```
1.3862943611198906` (n + 1) Log2[n] - 2.8455686701969345` n + 2.1544313298030655`
```

**Here is a series of plots of the exact solution (blue line) plotted against out approximation (purple line) and textbook's approximation (olive line):**

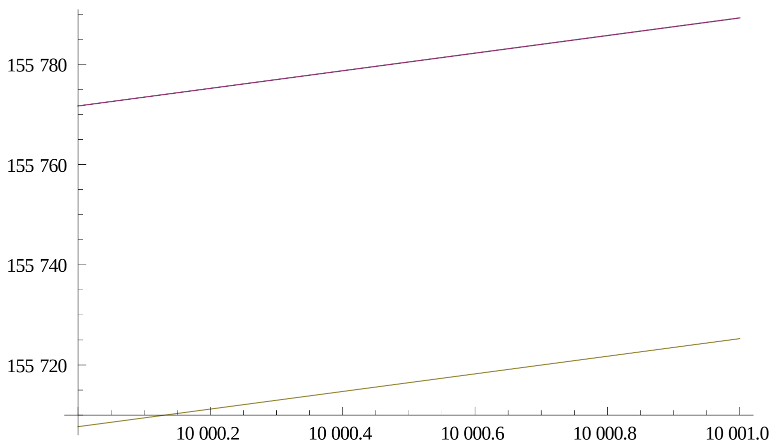
```
Plot[
 Tooltip[{2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]),
   1.3862943611198906` (n + 1) Log2[n] - 2.8455686701969345` n + 2.1544313298030655`,
   1.386 n Log2[n] - 2.846 n}], {n, 0, 5}]
```



```
Plot[
 Tooltip[{2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]),
   1.3862943611198906` (n + 1) Log2[n] - 2.8455686701969345` n + 2.1544313298030655`,
   1.386 n Log2[n] - 2.846 n}], {n, 5, 10}]
```




```
Plot[
 Tooltip[{2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]),
   1.3862943611198906` (n + 1) Log2[n] - 2.8455686701969345` n + 2.1544313298030655`,
   1.386 n Log2[n] - 2.846 n}], {n, 10000, 10001}]
```




**The rest of this file is optional –**
**recommended for the brightest and most diligent students.**

**Below is another approximation that does not use Euler's gamma but uses harmonic number, instead:**

**Simplify[(Log[4] n Log2[n] + 2 (-2 + EulerGamma) n + Log[4] Log2[n] + (1 + 2 EulerGamma))]**

(Log[2] - 4 n Log[2] + EulerGamma Log[4] + EulerGamma n Log[4] + (1 + n) Log[4] Log[n]) / Log[2]

**Simplify[(2 Log[2] n Log2[n] + 2 (-2 + EulerGamma) n + 2 Log[2] Log2[n] + (1 + 2 EulerGamma))]**

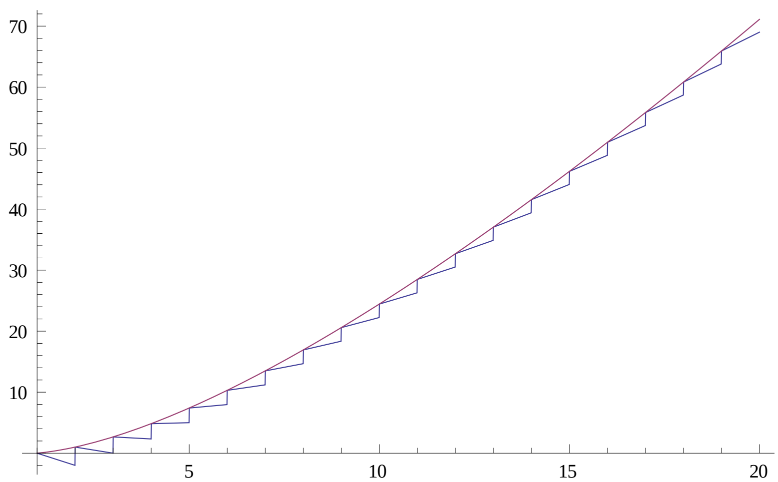
1 - 4 n + 2 EulerGamma (1 + n) + 2 (1 + n) Log[n]

**2 (n + 1) (Log[n] + EulerGamma) - 4 n + 1 ≈**

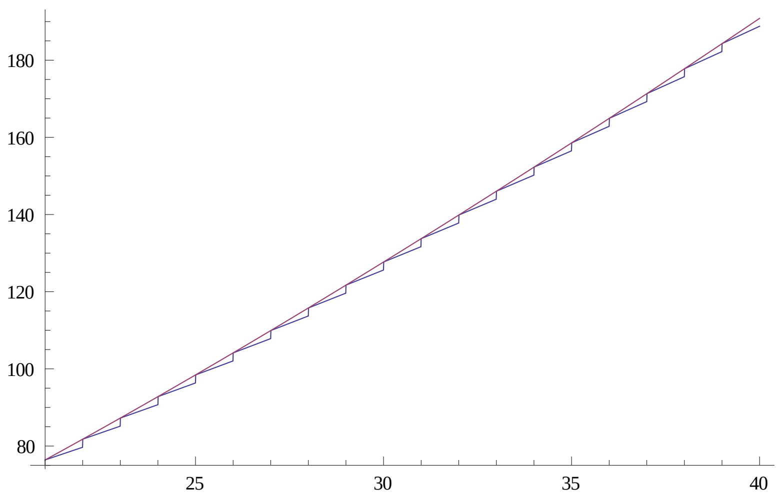
$$2\,(n+1)\left(\sum_{i=1}^{n}\frac{1}{i}\right)-4\,n$$

**The two coincide for integer arguments, as the graph below shows.**

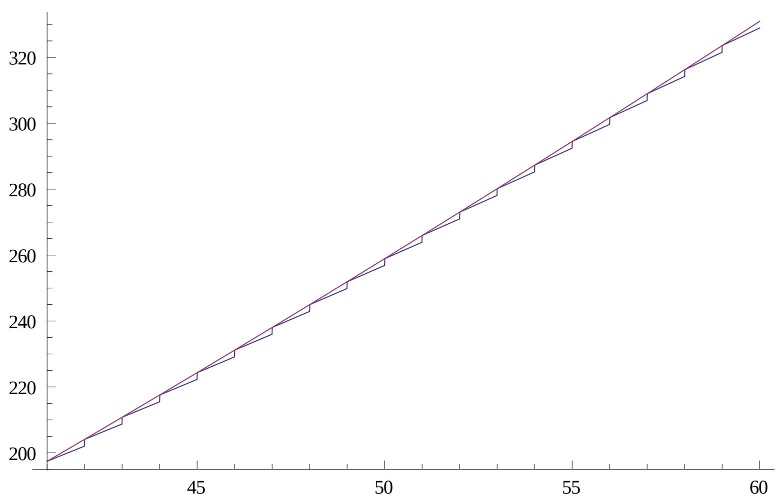
**Plot**$\left[\left\{2\,(n+1)\left(\sum_{i=1}^{n}\frac{1}{i}\right)-4\,n,\right.\right.$ **2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])}, {n, 1, 20}]**



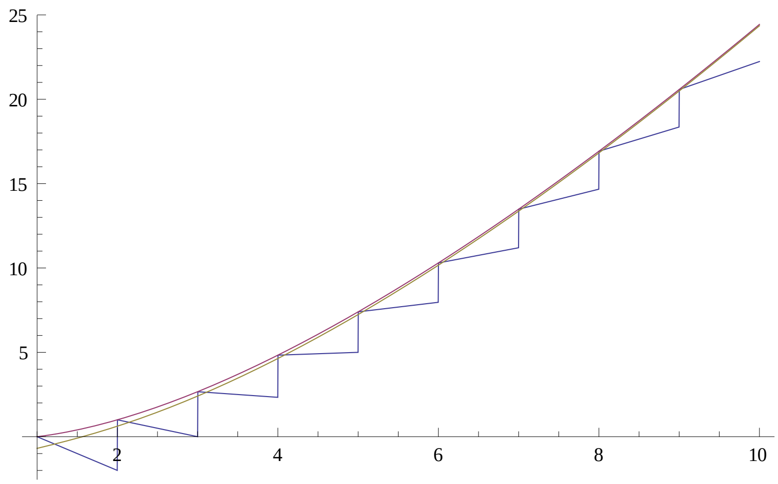
```
Plot[{2 (n + 1) (Sum[1/i, {i, 1, n}]) - 4 n,
  2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])}, {n,
  21, 40}]
```




```
Plot[{2 (n + 1) (Sum[1/i, {i, 1, n}]) - 4 n,
  2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])}, {n,
  41, 60}]
```



$$\text{Plot}\left[\left\{2\,(n+1)\left(\sum_{i=1}^{n}\frac{1}{i}\right)-4\,n,\right.\right.$$
$$2\,(\text{EulerGamma}-2\,n+\text{EulerGamma}\,n+\text{PolyGamma}[0,1+n]+n\,\text{PolyGamma}[0,1+n]),$$
$$\left.\left.2\,(n+1)\,(\text{Log}[n]+\text{EulerGamma})-4\,n+1\right\},\{n,1,10\}\right]$$




**Using**

**2 Log[n] = 2 Log[2] Log2[n]**

**N[2 Log[2]]**

1.38629

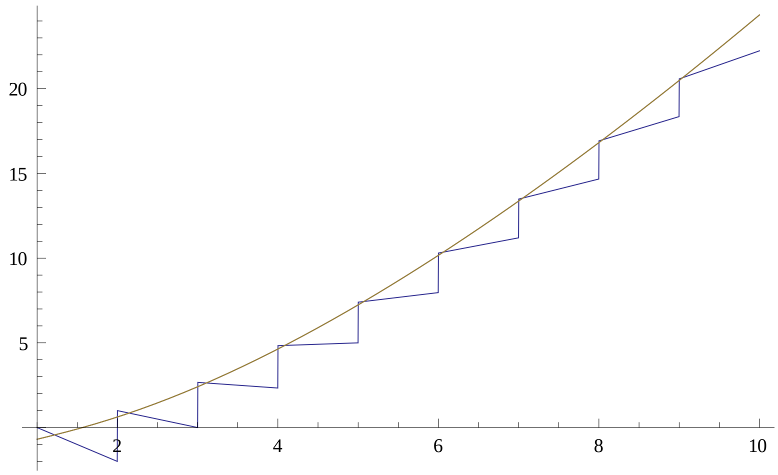
**we obtain :**

**A[n] ≈**

**1.3862943611198906` (n + 1) Log2[n] − 2.8455686701969345` n + 2.1544313298030655`**

```
Plot[{2 (n + 1) (Sum[1/i, {i, 1, n}]) - 4 n,
  1.3862943611198906` (n + 1) Log2[n] - 2.8455686701969345` n + 2.1544313298030655`,
  2 (n + 1) (Log[n] + EulerGamma) - 4 n + 1}, {n, 1, 10}]
```

$$\text{Plot}\left[\left\{2\,(n+1)\left(\sum_{i=1}^{n}\frac{1}{i}\right)-4\,n,\right.\right.$$




The above approximation was used as a reference in the demo program that counted number of comparisons done by QuickSort that I run for you in class.

Obviously, the approximation we computed (with a help from Mathematica) is Θ (n Log2[n]), because the limit of the ratio of the approximation to n Log2[n] is greater than 0 and less than ∞ :

```
Limit[{(1.3862943611198906` (n + 1) Log2[n] - 2.8455686701969345` n + 2.1544313298030655`) /
   (n Log2[n])}, n → ∞]
```

{1.38629}

Now, we will find an approximation of A[n] that converges to A[n] when n → ∞ faster than $\frac{1}{n}$ converges to 0 when n → ∞.

```
Limit[{n (2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]) -
     (Log[4] n Log2[n] + 2 (-2 + EulerGamma) n + Log[4] Log2[n] +
       (1 + (EulerGamma Log[16]) / Log[4])))}, n → ∞]
```

$\left\{\frac{5}{6}\right\}$

$$\text{Limit}\left[\left\{n\left(2\,(\text{EulerGamma} - 2\,n + \text{EulerGamma}\,n + \text{PolyGamma}[0, 1+n] + n\,\text{PolyGamma}[0, 1+n]) - \left(\text{Log}[4]\,n\,\text{Log2}[n] + 2\,(-2+\text{EulerGamma})\,n + \text{Log}[4]\,\text{Log2}[n] + (1 + (\text{EulerGamma}\,\text{Log}[16])/\text{Log}[4]) + \frac{5}{6\,n}\right)\right)\right\}, n \to \infty\right]$$

{0}

$$\text{Limit}\left[\left\{n\left(2\,(\text{EulerGamma} - 2\,n + \text{EulerGamma}\,n + \text{PolyGamma}[0, 1+n] + n\,\text{PolyGamma}[0, 1+n]) - \left(2\,\text{Log}[2]\,n\,\text{Log2}[n] + 2\,(-2+\text{EulerGamma})\,n + 2\,\text{Log}[2]\,\text{Log2}[n] + (1 + 2\,\text{EulerGamma}) + \frac{5}{6\,n}\right)\right)\right\}, n \to \infty\right]$$

{0}

$$\text{N}\left[2\,\text{Log}[2]\,n\,\text{Log2}[n] + 2\,(-2+\text{EulerGamma})\,n + 2\,\text{Log}[2]\,\text{Log2}[n] + (1 + 2\,\text{EulerGamma}) + \frac{5}{6\,n}\right]$$

$$2.15443 + \frac{0.833333}{n} - 2.84557\,n + 2.\,\text{Log}[n] + 2.\,n\,\text{Log}[n]$$

$$\text{N}\left[2\,\text{Log}[2]\,\text{Log2}[n]\,(n+1) + 2\,(-2+\text{EulerGamma})\,n + (1 + 2\,\text{EulerGamma}) + \frac{5}{6\,n}\right]$$

$$2.15443 + \frac{0.833333}{n} - 2.84557\,n + 2.\,(1. + n)\,\text{Log}[n]$$

N[5 / 6]

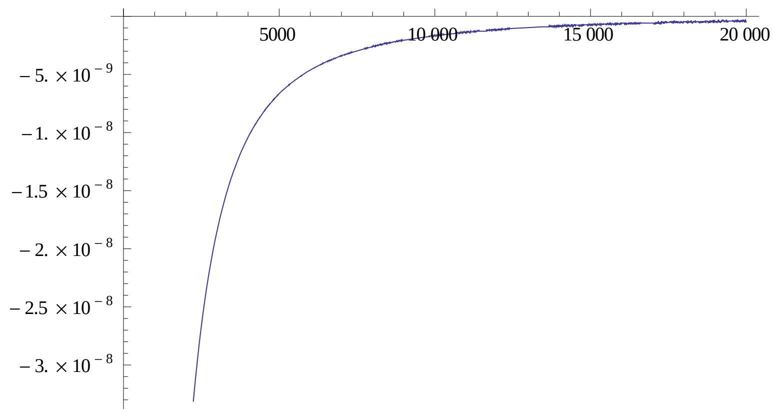
0.833333

**So,**

**A[n] ≈**

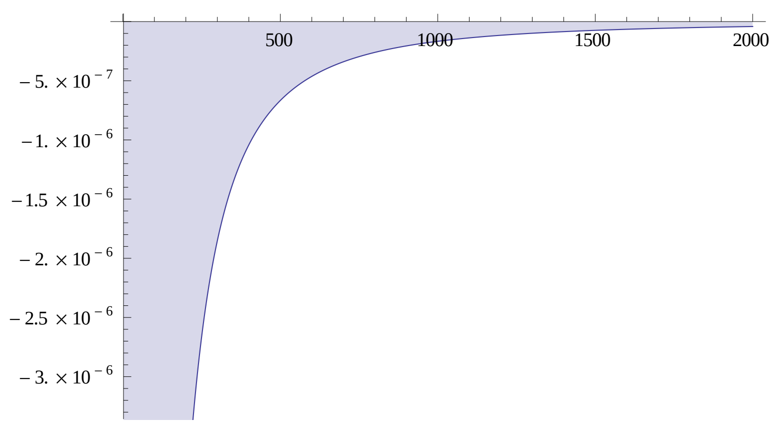
$$1.3862943611198906\text{`}\ (n+1)\,\text{Log2}[n] - 2.8455686701969345\text{`}\ n + 2.1544313298030655\text{`}\ + \frac{1}{n}\,0.8333333333333334\text{`}$$

```
Plot[(2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n]) -
    (Log[4] n Log2[n] + 2 (-2 + EulerGamma) n + Log[4] Log2[n] +
      (1 + (EulerGamma Log[16]) / Log[4]) + 5/(6 n))), {n, 2, 20000}]
```




**Here is comparision with exact solution that uses harmonic number :**

```
DiscretePlot[
 (2 (n + 1) (Sum[1/i, {i, 1, n}]) - 4 n - (Log[4] n Log2[n] + 2 (-2 + EulerGamma) n + Log[4] Log2[n] +
      (1 + (EulerGamma Log[16]) / Log[4]) + 5/(6 n))), {n, 2, 2000}]
```

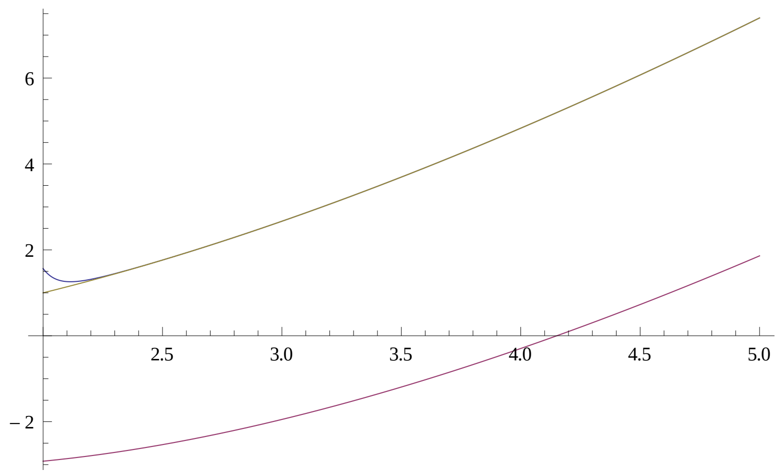



**Here is another approximation :**

**Limit**$\left[\left\{n^{32}\left(\left(\sum_{i=1}^{n}\frac{1}{i}\right)-\left(\text{Log}[n]+\text{EulerGamma}+\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}-\frac{1}{252n^6}+\frac{1}{240n^8}-\frac{1}{132n^{10}}+\frac{691}{32760n^{12}}-\frac{1}{12n^{14}}+\frac{3617}{8160n^{16}}-\frac{43867}{14364n^{18}}+\frac{174611}{6600n^{20}}-\frac{77683}{276n^{22}}+\frac{236364091}{65520n^{24}}-\frac{657931}{12n^{26}}+\frac{3392780147}{3480n^{28}}-\frac{1723168255201}{85932n^{30}}+\frac{7709321041217}{16320n^{32}}\right)\right)\right\}, n\to\infty\right]$

{0}

**Plot**$\left[\left\{2(n+1)\left(\text{Log}[n]+\text{EulerGamma}+\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}-\frac{1}{252n^6}+\frac{1}{240n^8}-\frac{1}{132n^{10}}+\frac{691}{32760n^{12}}-\frac{1}{12n^{14}}+\frac{3617}{8160n^{16}}-\frac{43867}{14364n^{18}}+\frac{174611}{6600n^{20}}-\frac{77683}{276n^{22}}+\frac{236364091}{65520n^{24}}-\frac{657931}{12n^{26}}+\frac{3392780147}{3480n^{28}}-\frac{1723168255201}{85932n^{30}}+\frac{7709321041217}{16320n^{32}}\right)-4n, 1.386\, n\, \text{Log2}[n]-2.846\, n, 2(\text{EulerGamma}-2n+\text{EulerGamma}\, n+\text{PolyGamma}[0,1+n]+n\, \text{PolyGamma}[0,1+n])\right\}, \{n,2,5\}\right]$




**Limit**$\left[\left\{n^{33}\left(2(n+1)\left(\text{Log}[n]+\text{EulerGamma}+\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}-\frac{1}{252n^6}+\frac{1}{240n^8}-\frac{1}{132n^{10}}+\frac{691}{32760n^{12}}-\frac{1}{12n^{14}}+\frac{3617}{8160n^{16}}-\frac{43867}{14364n^{18}}+\frac{174611}{6600n^{20}}-\frac{77683}{276n^{22}}+\frac{236364091}{65520n^{24}}-\frac{657931}{12n^{26}}+\frac{3392780147}{3480n^{28}}-\frac{1723168255201}{85932n^{30}}+\frac{7709321041217}{16320n^{32}}\right)-4n-2(\text{EulerGamma}-2n+\text{EulerGamma}\, n+\text{PolyGamma}[0,1+n]+n\, \text{PolyGamma}[0,1+n])\right)\right\}, n\to\infty\right]$

$\left\{\frac{151628697551}{6}\right\}$

So, here is a really close approximation with the error converging to 0 faster than $\frac{1}{n^{31}}$ :

$$A[n] \approx 2\,(n+1)\left(\text{Log}[n] + \text{EulerGamma} + \frac{1}{2\,n} - \frac{1}{12\,n^2} + \frac{1}{120\,n^4} - \frac{1}{252\,n^6} + \frac{1}{240\,n^8} - \frac{1}{132\,n^{10}} + \frac{691}{32760\,n^{12}} - \frac{1}{12\,n^{14}} + \frac{3617}{8160\,n^{16}} - \frac{43867}{14364\,n^{18}} + \frac{174611}{6600\,n^{20}} - \frac{77683}{276\,n^{22}} + \frac{236364091}{65520\,n^{24}} - \frac{657931}{12\,n^{26}} + \frac{3392780147}{3480\,n^{28}} - \frac{1723168255201}{85932\,n^{30}} + \frac{7709321041217}{16320\,n^{32}}\right) - 4\,n$$

$$\text{Expand}\left[2\,(n+1)\left(\text{Log}[n] + \text{EulerGamma} + \frac{1}{2\,n} - \frac{1}{12\,n^2}\right) - 4\,n\right]$$

$$1 + 2\,\text{EulerGamma} - \frac{1}{6\,n^2} + \frac{5}{6\,n} - 4\,n + 2\,\text{EulerGamma}\,n + 2\,\text{Log}[n] + 2\,n\,\text{Log}[n]$$

$$N\left[1 + 2\,\text{EulerGamma} - \frac{1}{6\,n^2} + \frac{5}{6\,n} - 4\,n + 2\,\text{EulerGamma}\,n + 2\,\text{Log}[n] + 2\,n\,\text{Log}[n]\right]$$

$$2.15443 - \frac{0.166667}{n^2} + \frac{0.833333}{n} - 2.84557\,n + 2.\,\text{Log}[n] + 2.\,n\,\text{Log}[n]$$

$$2.1544313298030655\text{`} - \frac{0.16666666666666666\text{`}}{n^2} + \frac{0.8333333333333334\text{`}}{n} - 2.8455686701969345\text{`}\,n + 2\,(n+1)\,n\,\text{Log}[n]$$

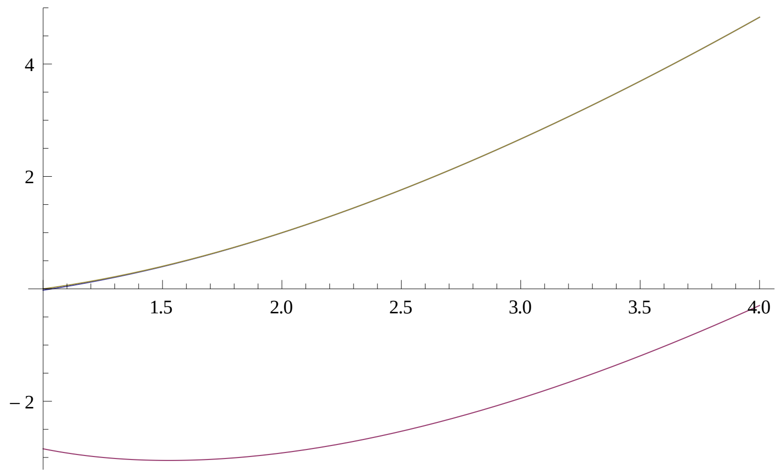
$1.3862943611198906\text{`}\,(n+1)\,\text{Log2}[n]$

$2.\text{`}\,(1+n)\,\text{Log}[n]$

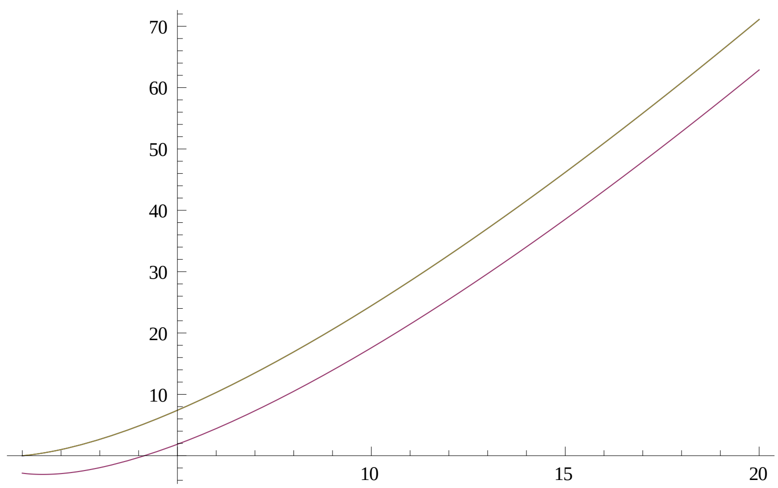
$A[n] \approx$

$$1.3862943611198906\text{`}\;(n+1)\,\text{Log2}[n] - 2.8455686701969345\text{`}\;n + 2.1544313298030655\text{`} + \frac{1}{n}\,0.8333333333333334\text{`} - \frac{0.16666666666666666\text{`}}{n^2}$$

```
Plot[Tooltip[{2.1544313298030655` - 0.16666666666666666` / n^2 +
    0.8333333333333334` / n - 2.8455686701969345` n + 2.` Log[n] + 2.` n Log[n],
   1.386 n Log2[n] - 2.846 n, 2 (EulerGamma - 2 n + EulerGamma n +
      PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])}], {n, 1, 4}]
```




```
Plot[Tooltip[{2.1544313298030655` - 0.16666666666666666` / n^2 +
    0.8333333333333334` / n - 2.8455686701969345` n + 2.` Log[n] + 2.` n Log[n],
   1.386 n Log2[n] - 2.846 n, 2 (EulerGamma - 2 n + EulerGamma n +
      PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])}], {n, 1, 20}]
```



```
Plot[Tooltip[{2.1544313298030655` - 0.16666666666666666` / n^2 + 0.8333333333333334` / n -
    2.8455686701969345` n + 2.` Log[n] + 2.` n Log[n], 1.386 n Log2[n] - 2.846 n,
   2 (EulerGamma - 2 n + EulerGamma n + PolyGamma[0, 1 + n] + n PolyGamma[0, 1 + n])}], {n,
  3000, 3001}]
```