# Pertains to Section 1.5 .4 The Asymptotic Order of Commonly Occurring Sums

Polynimial series

$$\sum_{i=1}^{n} i^{19}$$

$$\sum_{i=1}^{n} i^{d}$$

$$\int_{0}^{n} x^{d}\, dx \le \sum_{i=1}^{n} i^{d} \le \int_{1}^{n+1} x^{d}\, dx$$

$$\int_{0}^{n} x^{d}\, dx$$

$$\int_{1}^{n+1} x^{d}\, dx$$

$$\sum_{i=1}^{n} i^{d} \in \Theta\left(n^{d+1}\right)$$

Another trick (relates to figure 1.7)

For any non - decreasing function f,

$$\sum_{i=1}^{n} f(i) \leq n \times f(n)$$

and

$$\sum_{i=\frac{n}{2}}^{n} f(i) \leq \sum_{i=1}^{n} f(i)$$

Since

$$\sum_{i=\frac{n}{2}}^{n} f(i) \geq \frac{n}{2} \times f\left(\frac{n}{2}\right)$$

we conclude

$$\frac{n}{2} \times f\left(\frac{n}{2}\right) \leq \sum_{i=1}^{n} f(i) \leq n \times f(n)$$

Hence

$$\left(\frac{n}{2}\right)^{d+1} = \frac{n}{2} \times \left(\frac{n}{2}\right)^{d} \leq \sum_{i=1}^{n} i^{d} \leq n \times n^{d} = n^{d+1}$$

$\left(\frac{n}{2}\right)^{d+1} \in \Theta\left(n^{d+1}\right)$ and $n^{d+1} \in \Theta\left(n^{d+1}\right)$,

so

$$\sum_{i=1}^{n} i^{d} \in \Theta\left(n^{d+1}\right).$$

Geometric series

$$\sum_{i=0}^{n} 2^{i}$$

$$\sum_{i=0}^{n} \frac{1}{2^{i}}$$

$$\sum_{i=0}^{n} r^{i}$$

$$\sum_{i=0}^{n} r^{i} \in \Theta(r^{n}) \text{ if } r > 1$$

$\sum_{i=0}^{n} r^i \in \Theta(1)$ if $0 < r < 1$

Harmonic series (done before, quoted here for your convenience)

$$\sum_{i=1}^{n} \frac{1}{i}$$

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Plot[Tooltip[{∑_{i=1}^{n} 1/i, Log[n] + .577}], {n, 1, 20}]
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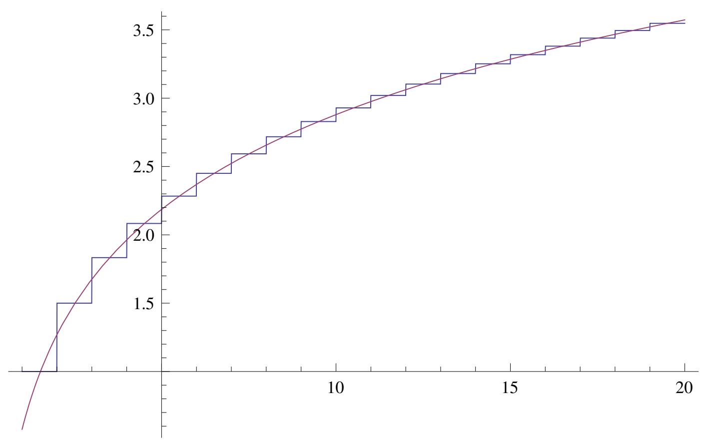


$$\int_2^{n+1} \frac{1}{x}\, dx \le \sum_{i=2}^{n} \frac{1}{i} \le \int_1^{n} \frac{1}{x}\, dx$$

$$\int_2^{n+1} \frac{1}{x}\, dx$$

Log[n + 1] - Log[2]

N[Log[2]]

$$\int_1^{n} \frac{1}{x}\, dx$$

Log[n]

So,

$$\text{Log}[n+1] - 0.694 \le \sum_{i=2}^{n} \frac{1}{i} \le \text{Log}[n]$$

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Plot[Tooltip[{Log[n], Sum[1/i, {i, 2, n}], Log[n + 1] - 0.694}], {n, 2, 20}]
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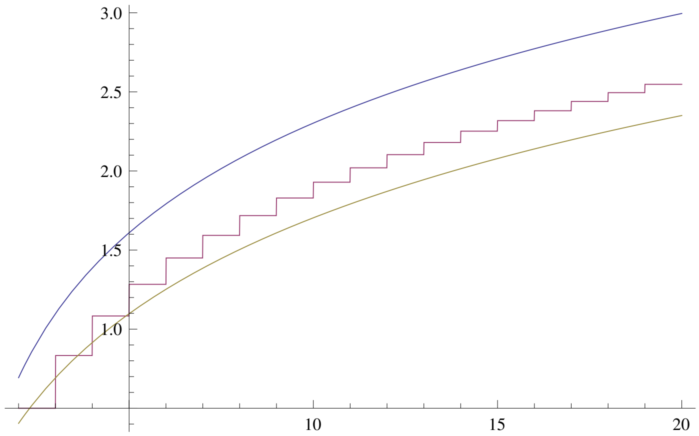
$\text{Plot}\left[\text{Tooltip}\left[\left\{\text{Log}[n],\ \sum_{i=2}^{n} \frac{1}{i},\ \text{Log}[n+1] - 0.694\right\}\right],\ \{n,\ 2,\ 20\}\right]$




$\sum_{i=1}^{n} \frac{1}{i} \in \Theta(\log n)$

Logarithmic series

$\frac{n}{2}(\text{Log2}[n] - 1) = \frac{n}{2}\text{Log2}\left[\frac{n}{2}\right] \leq \sum_{i=1}^{n} \text{Log2}[i] \leq n\,\text{Log2}[n]$

Since

$\frac{n}{2}(\text{Log2}[n] - 1) \in \Theta(n \log n)$

and

$n\,\text{Log2}[n] \in \Theta(n \log n)$

we conclude

$\sum_{i=1}^{n} \text{Log2}[i] \in \Theta(n \log n)$

## Exercise

Show that

$\sum_{i=1}^{n} i^d\,\text{Log2}[i] \in \Theta\left(n^{d+1} \log n\right)$