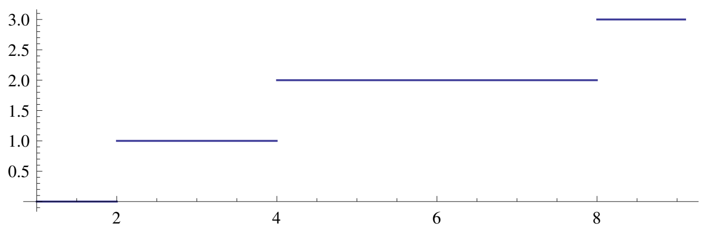
Clever computation of $\sum_{i=1}^{n} \lfloor \text{Log2}[i] \rfloor$ using experimental integration

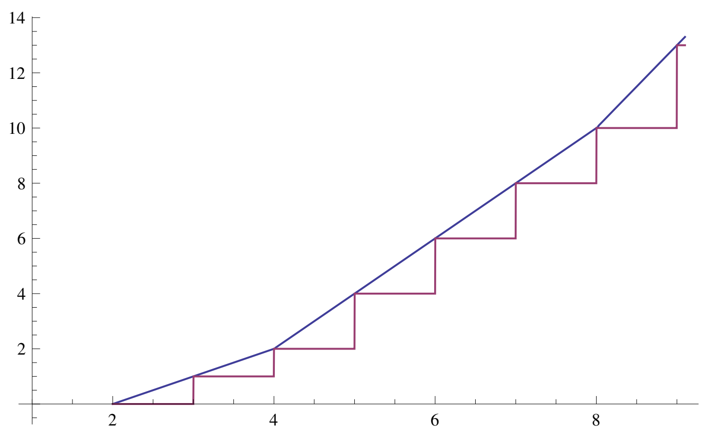
In[30]:= Plot[{⌊Log2[x]⌋}, {x, 1, 9.1}, AxesOrigin → {1, 0}, AspectRatio → .3, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]

Out[30]=



In[52]:= Plot$\left[\left\{\int_2^x \lfloor \text{Log2}[y] \rfloor \, dy,\ \sum_{i=2}^{\lfloor x \rfloor - 1} \lfloor \text{Log2}[i] \rfloor\right\}\right.$, {x, 2, 9.1}, AxesOrigin → {1, 0}, AspectRatio → .6, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]

Out[52]=



Thus for every integer x ≥ 2, $\int_2^x \lfloor \text{Log2}[y] \rfloor \, dy = \sum_{i=2}^{\lfloor x \rfloor - 1} \lfloor \text{Log2}[i] \rfloor$,

indeed.

1. Continuous solution

Since ⌊Log2[i]⌋ is constant between consecutive powers of 2,

symbolic computation of $\int_2^x \lfloor \text{Log2}[y] \rfloor \, dy$

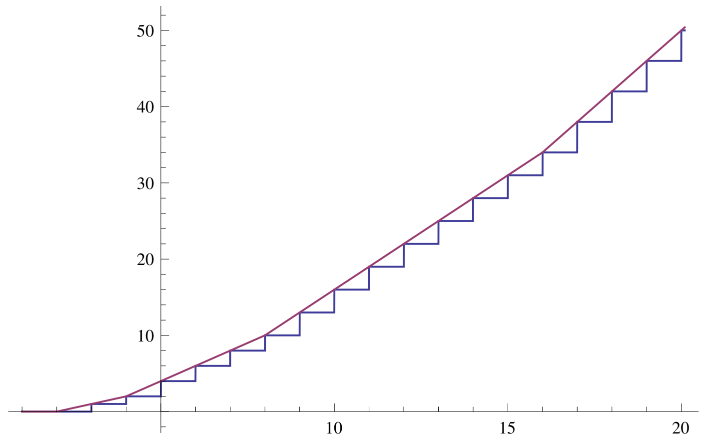
should be easy.

I am going to experimentally derive the following equalily :

$\sum_{i=2}^{\lfloor x \rfloor - 1} \lfloor \text{Log2}[i] \rfloor = x \lfloor \text{Log2}[x] \rfloor - \sum_{i=1}^{\lfloor \text{Log2}[x] \rfloor} 2^i$

In[32]:= $\text{Plot}\left[\left\{\sum_{i=2}^{\lfloor x\rfloor-1}\lfloor \text{Log2}[i]\rfloor,\ x\,\lfloor \text{Log2}[x]\rfloor-\sum_{i=1}^{\lfloor \text{Log2}[x]\rfloor}2^{i}\right\},\ \{x,\ 1,\ 20.1\},\ \text{PlotPoints}\to 200,\ \text{PlotTheme}\to \text{"Classic"},\ \text{PlotStyle}\to \{\text{Thickness}[\text{Medium}]\}\right]$

Out[32]=



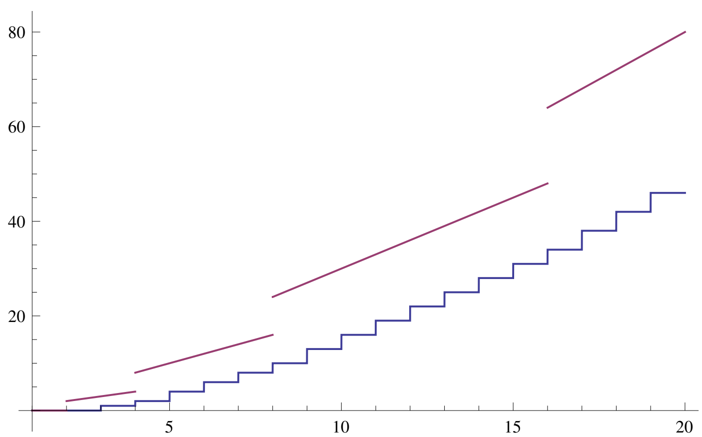
Since $\int c\, dx = c\,x + \text{constant}$,

let's try this

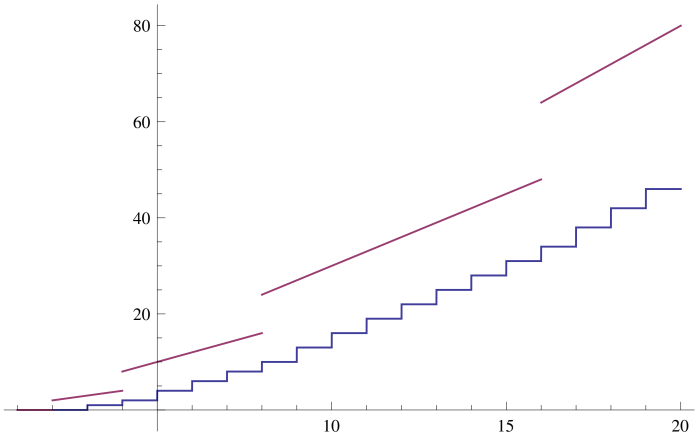
$\lfloor \text{Log2}[x]\rfloor\, x$ as a first guess for $\int_2^x \lfloor \text{Log2}[y]\rfloor\, dy$

In[53]:= $\text{Plot}\left[\left\{\sum_{i=2}^{\lfloor x\rfloor-1}\lfloor \text{Log2}[i]\rfloor\,(*\int_2^x\lfloor \text{Log2}[y]\rfloor dy*),\ \lfloor \text{Log2}[x]\rfloor\, x\right\},\ \{x,\ 1,\ 20\},\ \text{AxesOrigin}\to\{1,\ 0\},\ \text{PlotPoints}\to 200,\ \text{PlotTheme}\to \text{"Classic"},\ \text{PlotStyle}\to \{\text{Thickness}[\text{Medium}]\}\right]$

Out[53]=



Out[33]=



That was close, but not totally correct.

The crooked line (the top line)
$x \lfloor \text{Log2}[x] \rfloor$ jumps up by x for every
$2 \le x = 2^{\lfloor \text{Log2}[x] \rfloor}$.

This is so because when $2^k - 1 < x < 2^k$ then
$\lfloor \text{Log2}[x] \rfloor = k - 1$ and $x \lfloor \text{Log2}[x] \rfloor = x \times (k - 1)$, but when $x = 2^k$ then $x \lfloor \text{Log2}[x] \rfloor = x \times k$
so that the jump at $x = 2^k$ is
$x \times k - x \times (k - 1) = x$.

For example,
for $x = 2$ it jumps up by 2,
for $x = 4$ it jumps up by 4 (so the accumulated jump is $2 + 4 = 6$),
for $x = 8$ it jumps up by 8 (so the accumulated jump is $2 + 4 + 8 = 14$), etc.

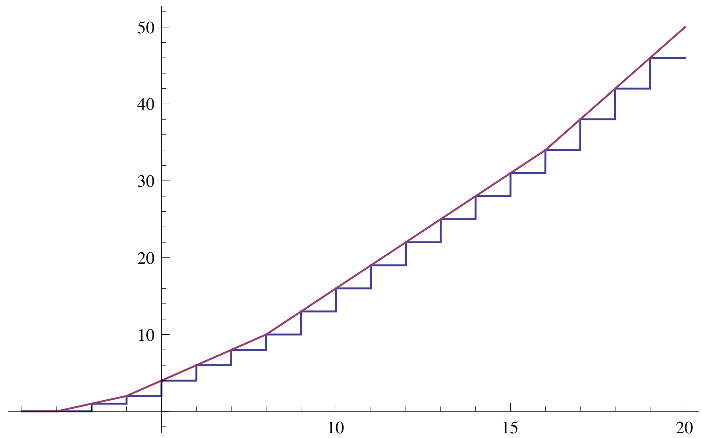
So, we need to subtract k from $x \lfloor \text{Log2}[x] \rfloor$
for every $2 \le k = 2^{\lfloor \text{Log2}[k] \rfloor} \le x$, that is,
(by applying $\lfloor \text{Log2}[\cdot] \rfloor$ to all sides of the above inequality and substituting
$i = \lfloor \text{Log2}[k] \rfloor$ so that $k = 2^i$),

subtract 2^i from $x \lfloor \text{Log2}[x] \rfloor$
for every $1 \le i \le \lfloor \text{Log2}[x] \rfloor$, that is,

subtract $\sum_{i=1}^{\lfloor \text{Log2}[x] \rfloor} 2^i$ from $x \lfloor \text{Log2}[x] \rfloor$

In[34]:= Plot[{$\sum_{i=1}^{\lfloor x\rfloor-1}$ ⌊Log2[i]⌋, x ⌊Log2[x]⌋ - $\sum_{i=1}^{\lfloor \text{Log2}[x]\rfloor} 2^i$}, {x, 1, 20},
PlotPoints → 200, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]

Out[34]=

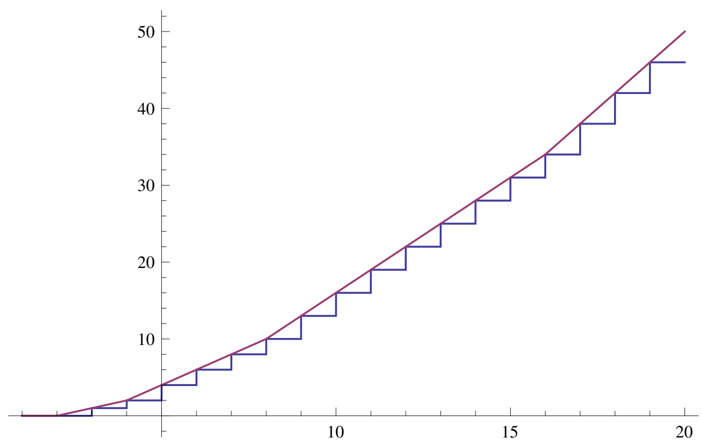


Bingo !

Now, we just simplify $\sum_{i=1}^{\lfloor \text{Log2}[x]\rfloor} 2^i$

In[35]:= Plot[{$\sum_{i=1}^{\lfloor x\rfloor-1}$ ⌊Log2[i]⌋, x ⌊Log2[x]⌋ - $\left(2^{\lfloor \text{Log2}[x]\rfloor+1} - 2\right)$}, {x, 1, 20},
PlotPoints → 200, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]

Out[35]=



Thus

$$\sum_{i=1}^{\lfloor x\rfloor-1} \lfloor \text{Log2}[i]\rfloor = x \lfloor \text{Log2}[x]\rfloor - 2^{\lfloor \text{Log2}[x]\rfloor+1} + 2$$

or putting x = n + 1 so that ⌊x⌋ - 1 = n,

$$\sum_{i=1}^{n} \lfloor \text{Log2}[i]\rfloor = (n+1) \lfloor \text{Log2}[n+1]\rfloor - 2^{\lfloor \text{Log2}[n+1]\rfloor+1} + 2$$

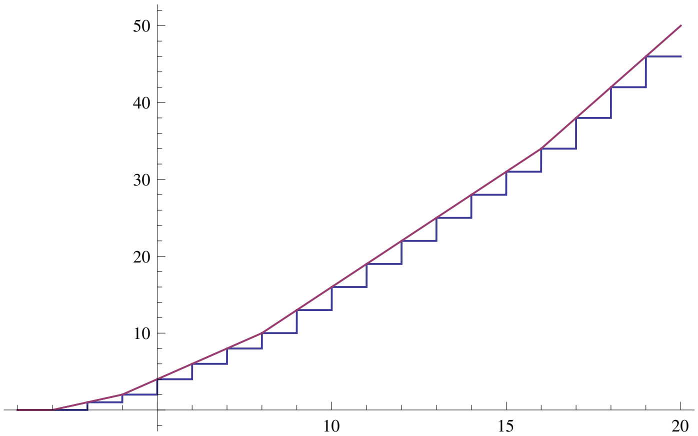
This complete the experimental derivation of the above formula

The rest of this file is optional for all students

A different (optional) way of getting there

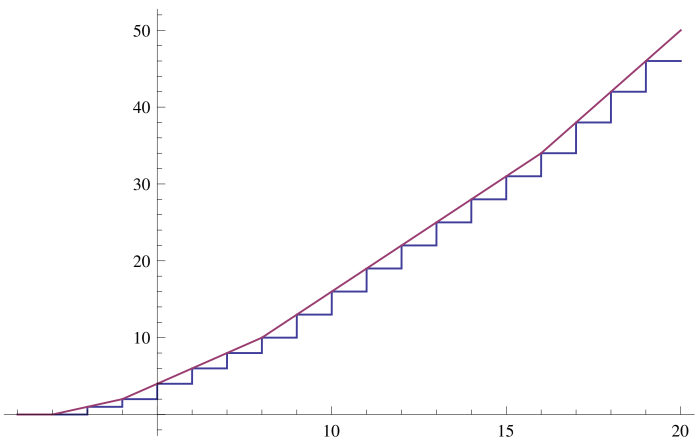
In[36]:= $\text{Plot}\left[\left\{\sum_{i=2}^{\lfloor x\rfloor} \lfloor \text{Log2}[i-1]\rfloor,\ x\,\lfloor \text{Log2}[x]\rfloor - \sum_{i=1}^{\lfloor \text{Log2}[x]\rfloor} 2^{i}\right\},\ \{x, 1, 20\},\right.$
$\left.\text{PlotPoints} \to 200,\ \text{PlotTheme} \to \text{"Classic"},\ \text{PlotStyle} \to \{\text{Thickness}[\text{Medium}]\}\right]$

Out[36]=



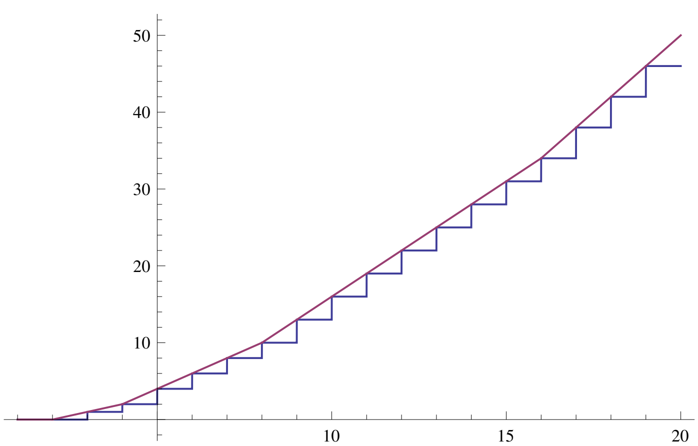
In[37]:= $\text{Plot}\left[\left\{\sum_{i=1}^{\lfloor x\rfloor-1} \lfloor \text{Log2}[i]\rfloor,\ x\,\lfloor \text{Log2}[x]\rfloor - \sum_{i=1}^{\lfloor \text{Log2}[x]\rfloor} 2^{i}\right\},\ \{x, 1, 20\},\right.$
$\left.\text{PlotPoints} \to 200,\ \text{PlotTheme} \to \text{"Classic"},\ \text{PlotStyle} \to \{\text{Thickness}[\text{Medium}]\}\right]$

Out[37]=



In[38]:= $\text{Plot}\left[\left\{\sum_{i=1}^{\lfloor x\rfloor-1} \lfloor \text{Log2}[i]\rfloor,\ x\,\lfloor \text{Log2}[x]\rfloor - \left(2^{\lfloor \text{Log2}[x]\rfloor+1} - 2\right)\right\},\ \{x, 1, 20\},\right.$
$\left.\text{PlotPoints} \to 200,\ \text{PlotTheme} \to \text{"Classic"},\ \text{PlotStyle} \to \{\text{Thickness}[\text{Medium}]\}\right]$

Out[38]=

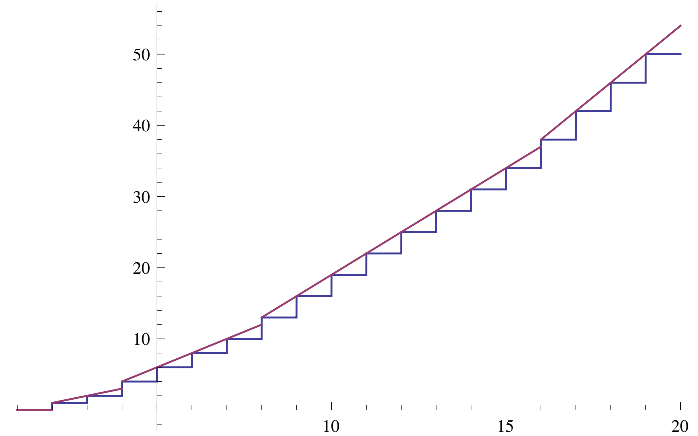


2. Discontinuous solution

$$\sum_{i=1}^{\lfloor x \rfloor} \lfloor \text{Log2}[i] \rfloor = x \lfloor \text{Log2}[x] \rfloor - \sum_{i=1}^{\lfloor \text{Log2}[x] \rfloor} \left(2^{i} - 1\right)$$

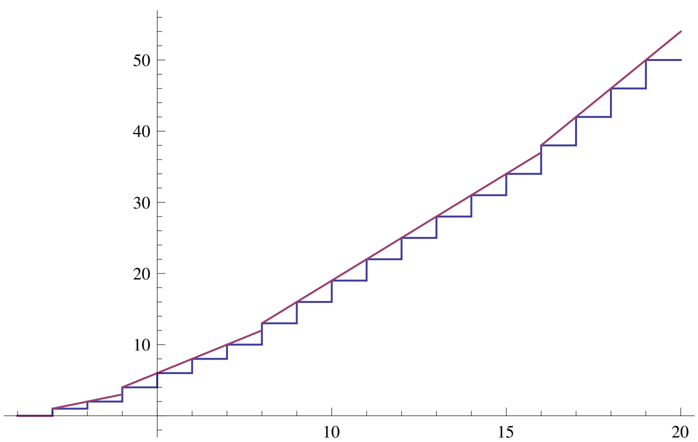
In[39]:= $\text{Plot}\left[\left\{\sum_{i=1}^{\lfloor x \rfloor} \lfloor \text{Log2}[i] \rfloor,\ x \lfloor \text{Log2}[x] \rfloor - \sum_{i=1}^{\lfloor \text{Log2}[x] \rfloor} \left(2^{i} - 1\right)\right\},\ \{x, 1, 20\},\right.$
$\left.\text{PlotPoints} \to 200,\ \text{PlotTheme} \to \text{"Classic"},\ \text{PlotStyle} \to \{\text{Thickness}[\text{Medium}]\}\right]$

Out[39]=



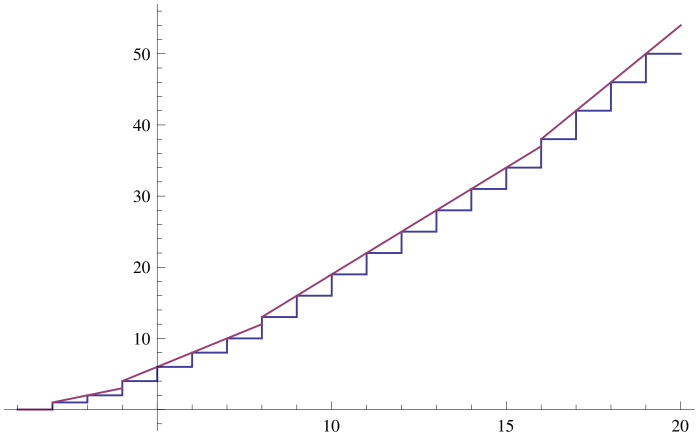
In[40]:= $\text{Plot}\left[\left\{\sum_{i=1}^{\lfloor x \rfloor} \lfloor \text{Log2}[i] \rfloor,\ x \lfloor \text{Log2}[x] \rfloor - \left(\sum_{i=1}^{\lfloor \text{Log2}[x] \rfloor} 2^{i} - \lfloor \text{Log2}[x] \rfloor\right)\right\},\ \{x, 1, 20\},\right.$
$\left.\text{PlotPoints} \to 200,\ \text{PlotTheme} \to \text{"Classic"},\ \text{PlotStyle} \to \{\text{Thickness}[\text{Medium}]\}\right]$

Out[40]=



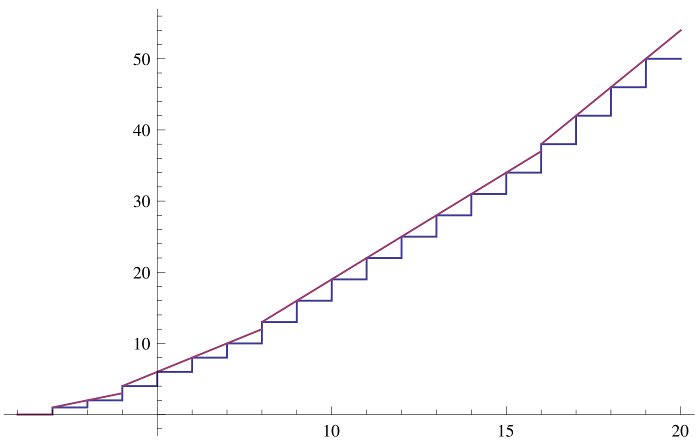
In[41]:=
```
Plot[{∑_{i=1}^{⌊x⌋} ⌊Log2[i]⌋, x ⌊Log2[x]⌋ - (2^(⌊Log2[x]⌋+1) - 2 - ⌊Log2[x]⌋)}, {x, 1, 20},
 PlotPoints → 200, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]
```

Out[41]=



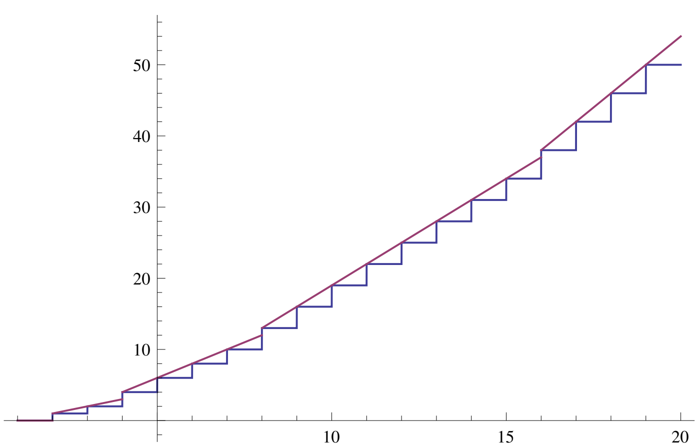
In[42]:=
```
Plot[{∑_{i=1}^{⌊x⌋} ⌊Log2[i]⌋, x ⌊Log2[x]⌋ - 2^(⌊Log2[x]⌋+1) + 2 + ⌊Log2[x]⌋}, {x, 1, 20},
 PlotPoints → 200, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]
```

Out[42]=



In[43]:=
```
Plot[{∑_{i=1}^{⌊x⌋} ⌊Log2[i]⌋, (x + 1) ⌊Log2[x]⌋ - 2^(⌊Log2[x]⌋+1) + 2}, {x, 1, 20},
 PlotPoints → 200, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]
```

Out[43]=



Thus we proved

$\sum_{i=1}^{\lfloor x \rfloor} \lfloor \text{Log2}[i] \rfloor = (x+1)\lfloor \text{Log2}[x] \rfloor - 2^{\lfloor \text{Log2}[x] \rfloor + 1} + 2$

or putting x = n

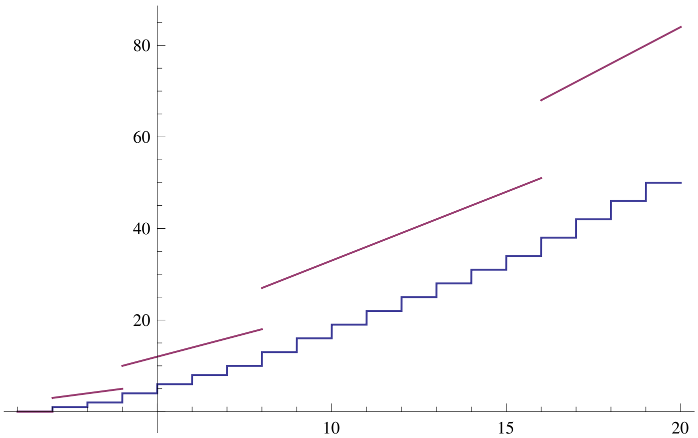
$\sum_{i=1}^{n} \lfloor \text{Log2}[i] \rfloor = (n+1)\lfloor \text{Log2}[n] \rfloor - 2^{\lfloor \text{Log2}[n] \rfloor + 1} + 2$

The continuous solution was

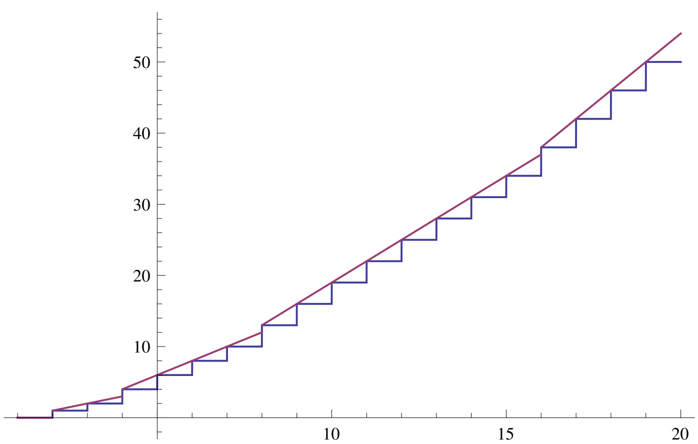
$\sum_{i=1}^{n} \lfloor \text{Log2}[i] \rfloor = (n+1)\lfloor \text{Log2}[n+1] \rfloor - 2^{\lfloor \text{Log2}[n+1] \rfloor + 1} + 2$

Another way to derive discontinuous solution

In[44]:= Plot$\left[\left\{\sum_{i=1}^{\lfloor x \rfloor} \lfloor \text{Log2}[i] \rfloor, (x+1)\lfloor \text{Log2}[x] \rfloor\right\}\right.$, {x, 1, 20},
PlotPoints → 200, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]

Out[44]=


In[45]:= Plot$\left[\left\{\sum_{i=1}^{\lfloor x \rfloor} \lfloor \text{Log2}[i] \rfloor, (x+1)\lfloor \text{Log2}[x] \rfloor - \sum_{i=1}^{\lfloor \text{Log2}[x] \rfloor} 2^i\right\}\right.$, {x, 1, 20},
PlotPoints → 200, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]

Out[45]=


In[46]:=
```
Plot[{∑_{i=1}^{⌊x⌋} ⌊Log2[i]⌋, (x + 1) ⌊Log2[x]⌋ - (2^(⌊Log2[x]⌋+1) - 2)}, {x, 1, 20},
 PlotPoints → 200, PlotTheme → "Classic", PlotStyle → {Thickness[Medium]}]
```

Out[46]=