Definitions of floor of and ceiling

$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid x \geq n\}$

$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid x \leq n\}$

where $\mathbb{Z}$ is the set of all integers.

Examples

$\lfloor 3 \rfloor = 3$
$\lfloor 3.14 \rfloor = 3$
$\lfloor -3.14 \rfloor = -4$

$\lceil 3 \rceil = 3$
$\lceil 3.14 \rceil = 4$
$\lceil -3.14 \rceil = -3$

$\lfloor 3 \rfloor$

$\lfloor 3.14 \rfloor$

$\lfloor -3.14 \rfloor$

$\lceil 3 \rceil$

$\lceil 3.14 \rceil$

$\lceil -3.14 \rceil$

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Plot[Tooltip[{⌊x⌋, x , ⌈x⌉}], {x, -5, 5}]
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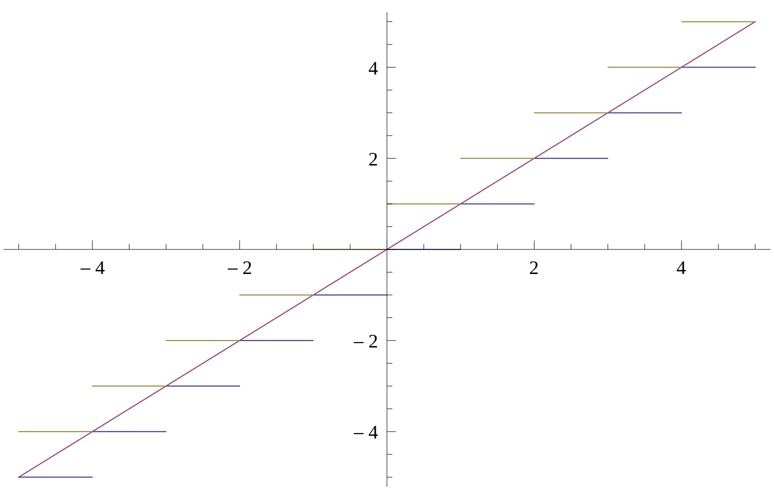


$\lfloor x \rfloor \le x \le \lceil x \rceil$

$(\forall n \in \mathbb{Z})\ (x \le n \Rightarrow \lceil x \rceil \le n)$

$(\forall n \in \mathbb{Z})\ (x \ge n \Rightarrow \lfloor x \rfloor \ge n)$

Relation between $\lfloor x \rfloor$ and $\lceil x \rceil$:

$\lfloor x \rfloor = -\lceil -x \rceil$

$\lceil x \rceil = -\lfloor -x \rfloor$

Exercise : Prove it !

$2^{\lceil \mathrm{Log2}[n] \rceil} = \min \{2^k \mid k \in \mathbb{N}\ \&\ n \le 2^k\}$

where $\mathbb{N}$ is the set {0, 1, 2, ...} of all natural numbers.

Proof.

$\lceil x \rceil = \min \{k \in \mathbb{Z} \mid x \le k\}$

$2^{\lceil x \rceil} = 2^{\min \{k \in \mathbb{Z} \mid x \le k\}} =$

[since $2^{\square}$ is a growing function]

$= \min \{2^k \mid k \in \mathbb{Z}\ \&\ 2^x \le 2^k\}$

Putting $x = \mathrm{Log2}[n]$ we get

$2^{\lceil \mathrm{Log2}[n] \rceil} = \min \{2^k \mid k \in \mathbb{Z}\ \&\ 2^{\mathrm{Log2}[n]} \le 2^k\} =$

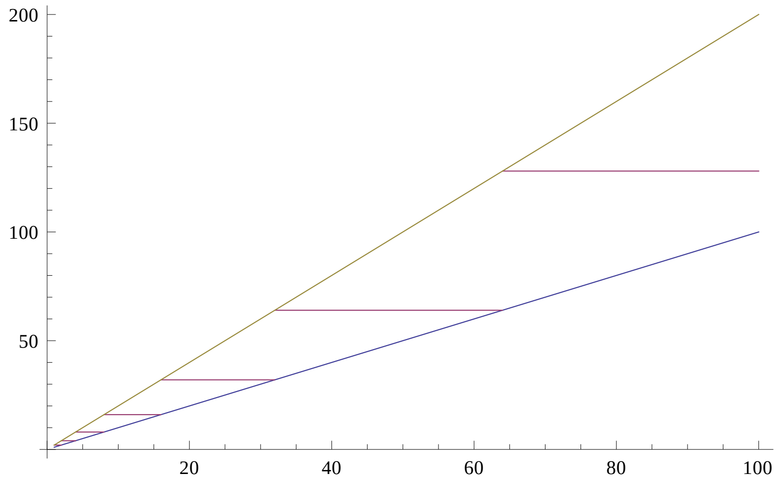
$\min \{2^k \mid k \in \mathbb{Z}\ \&\ n \le 2^k\}$ = [since $n > 0$]

$\min \{2^k \mid k \in \mathbb{N}\ \&\ n \le 2^k\}$ □

$n \le 2^{\lceil \mathrm{Log2}[n] \rceil} < 2n$

Exercise : Prove it !

$\text{Plot}\left[\text{Tooltip}\left[\left\{n,\ 2^{\lceil \text{Log2}[n] \rceil},\ 2\,n\right\}\right],\ \{n,\ 1,\ 100\}\right]$

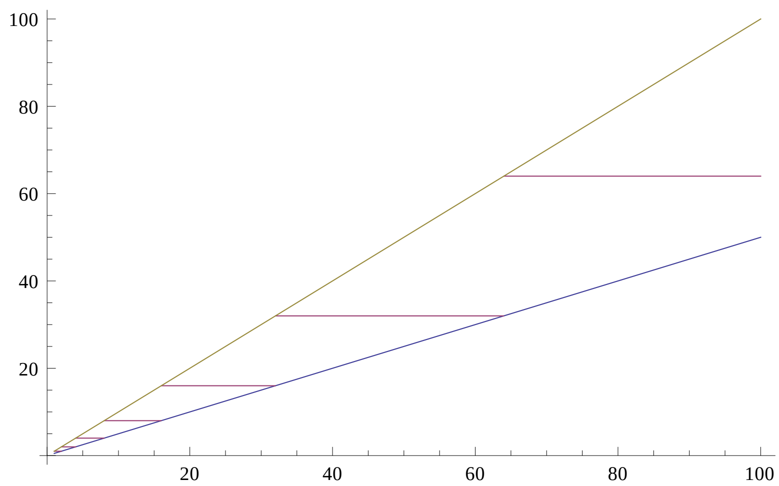


$2^{\lfloor \text{Log2}[n] \rfloor} = \max\left\{2^k \mid k \in \mathbb{N}\ \&\ n \geq 2^k\right\}$

Exercise : Prove it !

$$\frac{n}{2} < 2^{\lfloor \text{Log2}[n] \rfloor} \leq n$$

$\text{Plot}\left[\text{Tooltip}\left[\left\{\frac{n}{2},\ 2^{\lfloor \text{Log2}[n] \rfloor},\ n\right\}\right],\ \{n,\ 1,\ 100\}\right]$



The number of bits necessary to represent a positive integer n is

$\lfloor \text{Log2}[n] \rfloor + 1$

Examples

12 = 1100 (in binary, 4 bits)

$\lfloor \text{Log2}[12] \rfloor + 1$

1 234 567 891 = 1 001 001 100 101 100 000 001 011 010 011
(in binary, 31 bits)

$\lfloor \text{Log2}[1\,234\,567\,891] \rfloor + 1$

$\lfloor \text{Log2}[n] \rfloor + 1 = \lceil \text{Log2}[n+1] \rceil$

Proof in the next file.