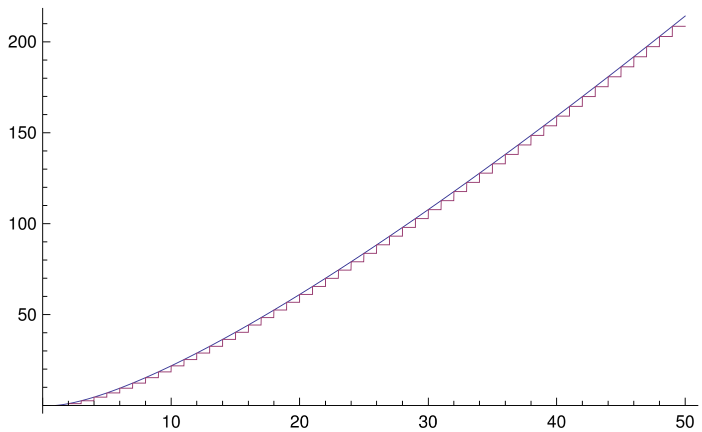
Experimental computation of :

$\text{Log2}[n!] \approx \left(n + \frac{1}{2}\right) \text{Log2}[n] - 1.44\, n + 1.33$

More exactly,

$$\text{Log2}[n!] = \sum_{i=1}^{n} \text{Log2}[i] \approx n\,\text{Log2}[n] - \frac{1}{\text{Log}[2]} n + \frac{\text{Log2}[n]}{2} + \frac{\text{Log}[2\pi]}{\text{Log}[4]} + 1/(\text{Log}[4096]\, n) - 1\Big/\left(360\,\text{Log}[2]\, n^3\right) \pm o\left(\frac{1}{n^3}\right)$$

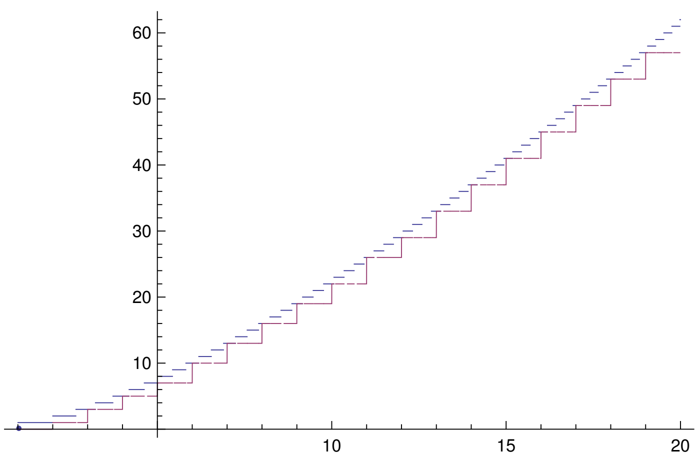
$$\text{Plot}\left[\left\{n\,\text{Log2}[n] - \text{Log2}[e]\, n + \frac{\text{Log2}[n]}{2} + \frac{1}{2}\text{Log2}[2\pi] + \frac{\text{Log2}[e]}{12\, n} - 1\Big/\left(360\,\text{Log}[2]\, n^3\right), \sum_{i=1}^{n} \text{Log2}[i]\right\}, \{n, 1, 50\}, \text{PlotPoints} \to 100\right]$$



Perfect match !

Here is a test of the above approximation for computing the exact value of $\lceil \text{Log2}[n!] \rceil$

$$\text{Plot}\Big[\text{Tooltip}\Big[\left\{\left\lceil n\,\text{Log2}[n] - \text{Log2}[e]\, n + \frac{1}{2}\text{Log2}[n] + \frac{1}{2}\text{Log2}[2\pi] + \text{Log2}[e]/(12\, n) - 1\Big/\left(360\,\text{Log}[2]\, n^3\right)\right\rceil, \left\lceil \sum_{i=1}^{n} \text{Log2}[i] \right\rceil\right\}\Big], \{n, 1, 20\}, \text{PlotPoints} \to 200\Big]$$



It works up to fairly large values of n.

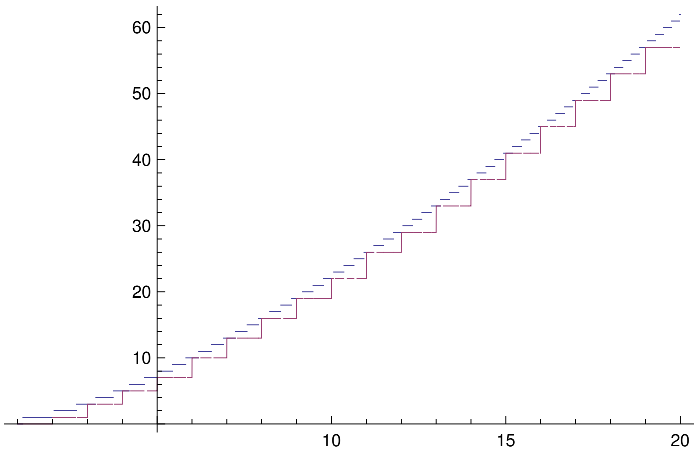
```
(*ProjectExercise:
 Write a program that searches for n that makes the approximation not
  equal to ⌈Log2[n!]⌉ and let me know such n if  your program found it*)
```

A more practical lower bound on ⌈Log2[n!]⌉ that uses

$$\sum_{i=1}^{n} \text{Log2}[i] \approx \left(n+\frac{1}{2}\right) \text{Log2}[n] - 1.4426950408889634\grave{}\, n + 1.3257480647361592\grave{} + (0.12022458674074696\grave{}/n) - (0.004007486224691565\grave{}/n^3)$$

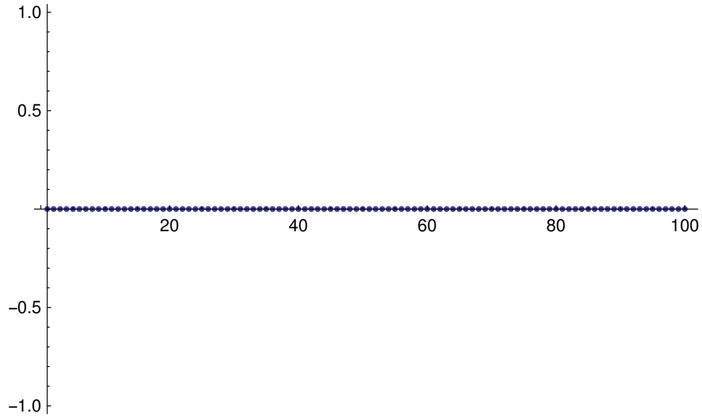
:

$$\text{Plot}\left[\text{Tooltip}\left[\left\{\left\lceil\left(n+\frac{1}{2}\right)\text{Log2}[n] - 1.4426950408889634\, n + 1.3257480647361592\right\rceil, \left\lceil\sum_{i=1}^{n}\text{Log2}[i]\right\rceil\right\}\right], \{n, 1, 20\}, \text{PlotPoints} \rightarrow 200\right]$$



Testing the difference between the two -
 if it is 0 or less then the "approximate" lower bound is correct.

```
DiscretePlot[
 {⌈(n + 1/2) Log2[n] - 1.4426950408889634 n + 1.3257480647361592⌉ - ⌈∑_{i=1}^{n} Log2[i]⌉},
 {n, 1, 100}]
```



Using our most exact approximation of Log2[n!], one can show that for n > 100,

$$\left(n + \frac{1}{2}\right) \text{Log2}[n] - 1.4426950408889634\, n + 1.3257480647361592 < \sum_{i=1}^{n} \text{Log2}[i]$$

This yields a slightly lesser but simpler lower bound of $\sum_{i=1}^{n} \text{Log2}[i]$

So, our simpler lower bound

$$\left\lceil \left(n + \frac{1}{2}\right) \text{Log2}[n] - 1.4426950408889634\, n + 1.3257480647361592 \right\rceil$$

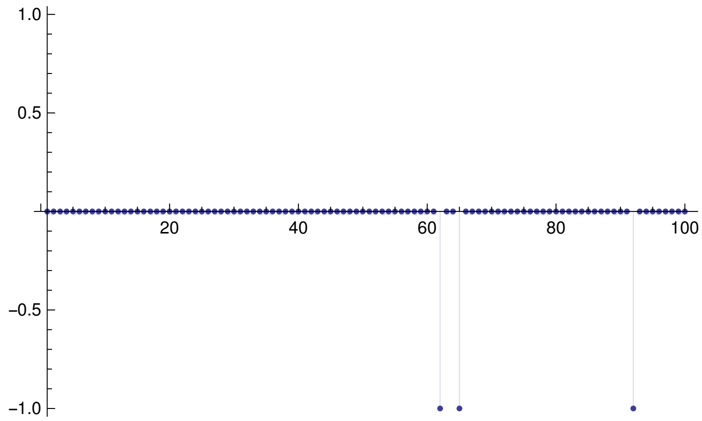
of ⌈Log2[n!]⌉ is a correct lower bound for all n.

However, this simpler lower bound slightly underestimates ⌈Log2[n!]⌉.

So, although it is a valid lower bound of ⌈Log2[n!]⌉,
it is not always equal to ⌈Log2[n!]⌉.

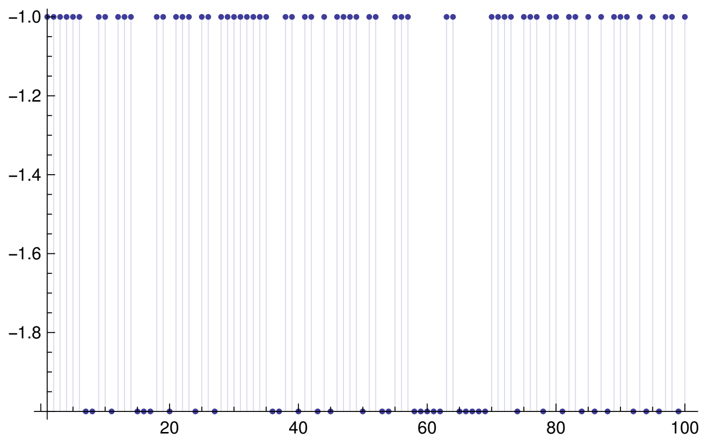
$$\left\lceil \left(n + \frac{1}{2}\right) \text{Log2}[n] - 1.443\, n + 1.325 \right\rceil$$

```
DiscretePlot[{[(n + 1/2) Log2[n] - 1.443 n + 1.325] - [∑_{i=1}^{n} Log2[i]]}, {n, 1, 100}]
```



Here is a test of the textbook approximation

```
DiscretePlot[{[(n + 1/2) Log2[n] - 1.443 n] - [∑_{i=1}^{n} Log2[i]]}, {n, 1, 100}]
```



Not nearly as good as ours !

$$\sum_{i=1}^{n} \text{Log2}[i] \approx$$

$$n\,\text{Log2}[n] - \text{Log2}[e]\,n + \frac{1}{2}\text{Log2}[n] +$$

$$\frac{1}{2}\text{Log2}[2\pi] + \text{Log2}[e]/(12\,n) - 1/(360\,\text{Log}[2]\,n^3)$$

and the difference between

this and the exact value is in $o\left(\frac{1}{n^3}\right)$.

Sterling formula (see file Mathematica / Stirling_formula.nb) would yield this :

$$\sum_{i=1}^{n} \text{Log2}[i] =$$

$$\text{Log2}[n!] > \text{Log2}\left[\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right] > \left(n+\frac{1}{2}\right)\text{Log2}[n] - \frac{n}{\text{Log}[2]} + \frac{1}{2}\text{Log2}[2\pi] \approx$$

$$\left(n+\frac{1}{2}\right)\text{Log2}[n] - 1.4426950408889634\text{`}\, n + 1.3257480647361592\text{`} >$$

$$\left(n+\frac{1}{2}\right)\text{Log2}[n] - 1.443\, n + 1.325$$

and

$$\sum_{i=1}^{n} \text{Log2}[i] = \text{Log2}[n!] < \text{Log2}\left[\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+\frac{1}{11\, n}\right)\right] <$$

$$\left(n+\frac{1}{2}\right)\text{Log2}[n] - \frac{n}{\text{Log}[2]} + \frac{1}{2}\text{Log2}[2\pi] + \text{Log2}\left[1+\frac{1}{11\, n}\right] \approx$$

$$\left(n+\frac{1}{2}\right)\text{Log2}[n] - 1.4426950408889634\text{`}\, n + 1.3257480647361592\text{`} + \text{Log2}\left[1+\frac{1}{11\, n}\right] <$$

$$\left(n+\frac{1}{2}\right)\text{Log2}[n] - 1.442\, n + 1.326$$

So, again,

$$\sum_{i=1}^{n} \text{Log2}[i] \approx \left(n+\frac{1}{2}\right)\text{Log2}[n] - 1.443\, n + 1.326$$

Either experimental or Stirling formula method is OK for derivation of $\sum_{i=1}^{n} \text{Log2}[i] \approx \left(n+\frac{1}{2}\right)\text{Log2}[n] - 1.443\, n + 1.326$

All computations below are optional for all students.

Simplifications:

$$\frac{1}{\text{Log}[2]} = \text{Log2}[e]$$

$$\frac{\text{Log2}[n]}{2} = \text{same}$$

$$\frac{\text{Log}[2\,\pi]}{\text{Log}[4]} = (\text{Log}[2] + \text{Log}[\pi]) \,/\, (2\,\text{Log}[2]) = \frac{1}{2}\,(1 + \text{Log2}[\pi])$$

$$\frac{1}{\text{Log}[4096]} = \frac{1}{12\,\text{Log}[2]} = \frac{\text{Log2}[e]}{12}$$

Ckeck:

$$N\left[\frac{1}{\text{Log}[2]}\right]$$

1.4427

N[Log2[e]]

1.4427

$$N\left[\frac{\text{Log}[2\,\pi]}{\text{Log}[4]}\right]$$

1.32575

$$N\left[\frac{1 + \text{Log2}[\pi]}{2}\right]$$

1.32575

$$N\left[\frac{\text{Log2}[2\,\pi]}{2}\right]$$

1.32575

```
N[1/Log[4096]]
```

0.120225

```
N[Log2[e]/12]
```

0.120225

```
N[1/(360 Log[2])]
```

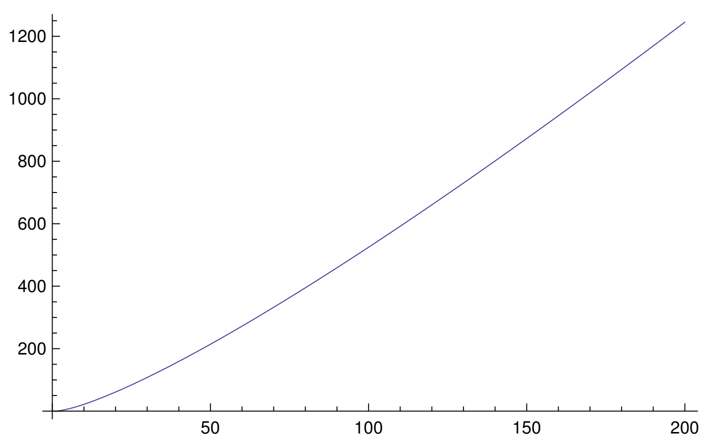
0.00400749

OK

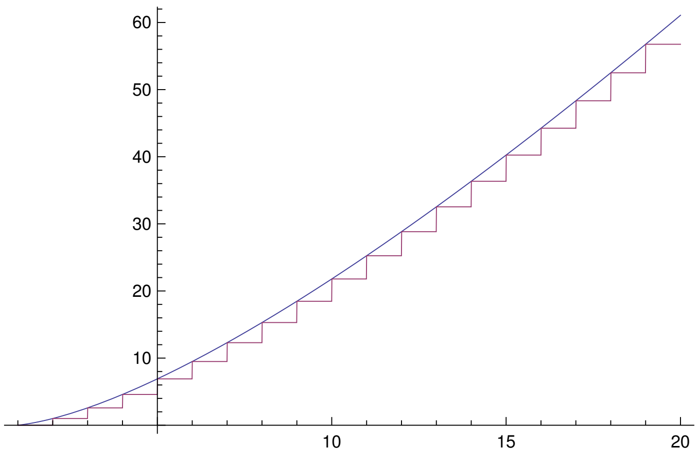
$$\sum_{i=1}^{n} \text{Log2}[i] \approx$$

$$n\,\text{Log2}[n] - 1.4426950408889634\grave{}\ n + \frac{\text{Log2}[n]}{2} + 1.3257480647361592\grave{} + 0.12022458674074696\grave{} / n - 0.004007486224691565\grave{} / n^3$$

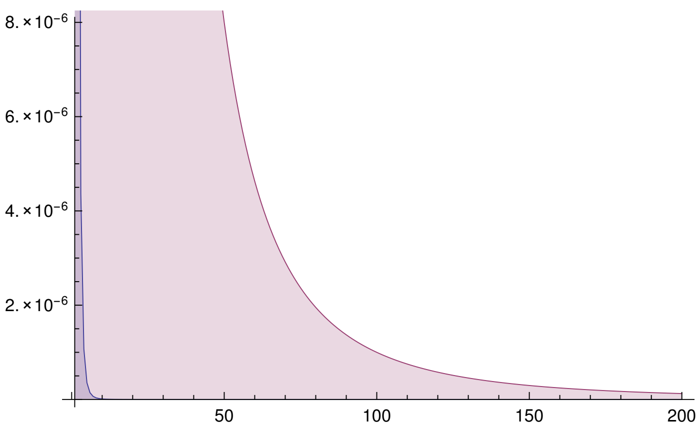
$$\text{Plot}\left[n\,\text{Log2}[n] - \text{Log2}[e]\,n + \frac{\text{Log2}[n]}{2} + \frac{\text{Log2}[2\pi]}{2} + \frac{\text{Log2}[e]}{12\,n} - 1 / \left(360\,\text{Log}[2]\,n^3\right),\ \{n, 1, 200\}\right]$$



```
Plot[{n Log2[n] - Log2[e] n + Log2[n]/2 + 1/2 Log2[ 2 π] + Log2[e]/(12 n) - 1/(360 Log[2] n^3),
  Sum[Log2[i], {i, 1, n}]}, {n, 1, 20}]
```



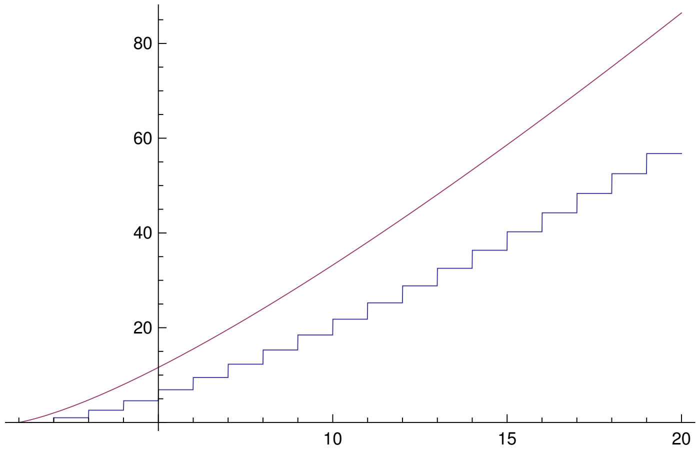

```
DiscretePlot[
 {Sum[Log2[i], {i, 1, n}] - (n Log2[n] - Log2[e] n + 1/2 Log2[n] + 1/2 Log2[ 2 π] + Log2[e] / (12 n) -
      1/(360 Log[2] n^3)), 1/n^3}, {n, 1, 200}]
```



Here is the computation of the above result :

```
Limit[{(Sum[Log2[i], {i, 1, n}]) / (n Log2[n])}, n → ∞]
{1}
```

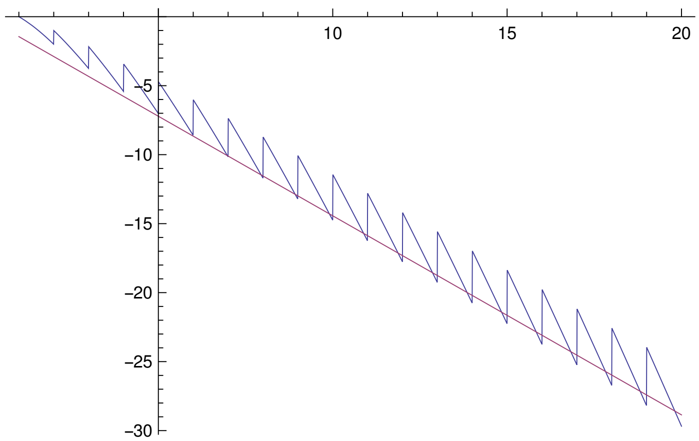
```
Plot[Tooltip[{∑_{i=1}^{n} Log2[i], n Log2[n]}], {n, 1, 20}]
```




```
Limit[{(∑_{i=1}^{n} Log2[i]) - (n Log2[n])}, n → ∞]
```

{-∞}

```
Plot[Tooltip[{(∑_{i=1}^{n} Log2[i]) - (n Log2[n]), -1.443 n}], {n, 1, 20}]
```



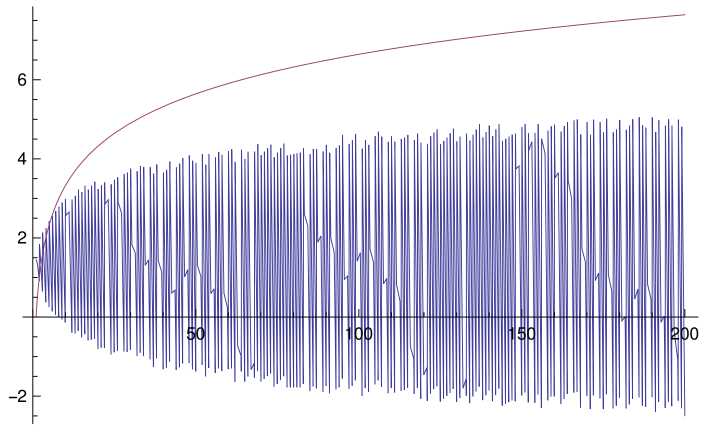

```
Limit[{((∑_{i=1}^{n} Log2[i]) - (n Log2[n])) / n}, n → ∞]
```

$\left\{-\frac{1}{\text{Log}[2]}\right\}$

```
Limit[{((∑_{i=1}^{n} Log2[i]) - ((n Log2[n]) - (1 / Log[2]) n))}, n → ∞]
```

{∞}

```
Plot[Tooltip[{((∑_{i=1}^{n} Log2[i]) - ((n Log2[n]) - (1 / Log[2]) n)), Log2[n]}], {n, 1, 200}]
```




```
Limit[{((∑_{i=1}^{n} Log2[i]) - ((n Log2[n]) - (1 / Log[2]) n)) / Log2[n]}, n → ∞]
```

$\left\{\frac{\text{Log}[2]}{\text{Log}[4]}\right\}$

```
Limit[{((∑_{i=1}^{n} Log2[i]) - ((n Log2[n]) - (1 / Log[2]) n)) / (Log2[n] / 2)}, n → ∞]
```

{1}

```
Limit[
 {((∑_{i=1}^{n} Log2[i]) - ((n Log2[n]) - (1 / Log[2]) n + 1/2 Log2[n] + Log[2 π] / Log[4]))}, n → ∞]
```

{0}

```
Limit[
 {n ((∑_{i=1}^{n} Log2[i]) - ((n Log2[n]) - (1 / Log[2]) n + 1/2 Log2[n] + Log[2 π] / Log[4]))}, n → ∞]
```

$\left\{\frac{1}{\text{Log}[4096]}\right\}$

```
Limit[{n ((∑_{i=1}^{n} Log2[i]) - ((n Log2[n]) - (1 / Log[2]) n +
        1/2 Log2[n] + Log[2 π] / Log[4] + 1 / (Log[4096] n)))}, n → ∞]
```

{0}

```
Limit[{n^2 ((∑_{i=1}^{n} Log2[i]) - ((n Log2[n]) - (1 / Log[2]) n +
        1/2 Log2[n] + Log[2 π] / Log[4] + 1 / (Log[4096] n)))}, n → ∞]
```

{0}

```
Limit[{n^3 ((Sum[Log2[i], {i, 1, n}]) - ((n Log2[n]) - (1 / Log[2]) n +
      1/2 Log2[n] + Log[2 π] / Log[4] + 1 / (Log[4096] n)))}, n → ∞]
```

$$\left\{-\frac{1}{360\,\text{Log}[2]}\right\}$$

```
Limit[{n^3 ((Sum[Log2[i], {i, 1, n}]) - (n Log2[n] - (1 / Log[2]) n + 1/2 Log2[n] +
      Log[2 π] / Log[4] + 1 / (Log[4096] n) - 1 / (360 Log[2] n^3)))}, n → ∞]
```

{0}

```
DiscretePlot[
 {((Sum[Log2[i], {i, 1, n}]) - (n Log2[n] - (1 / Log[2]) n + 1/2 Log2[n] + Log[2 π] / Log[4] + 1 /
      (Log[4096] n) - 1 / (360 Log[2] n^3)))}, {n, 1000, 1200}]
```