A proof of

$$(n+1)\lfloor \mathrm{Log2}[n+1]\rfloor - 2^{\lfloor \mathrm{Log2}[n+1]\rfloor+1} + 2 = (n+1)\,(\mathrm{Log2}[n+1] + \epsilon[n+1]) - 2n$$

where ϵ is given by the following definition

In[2]:=
```
β[x_] := 1 + x - 2^x
```

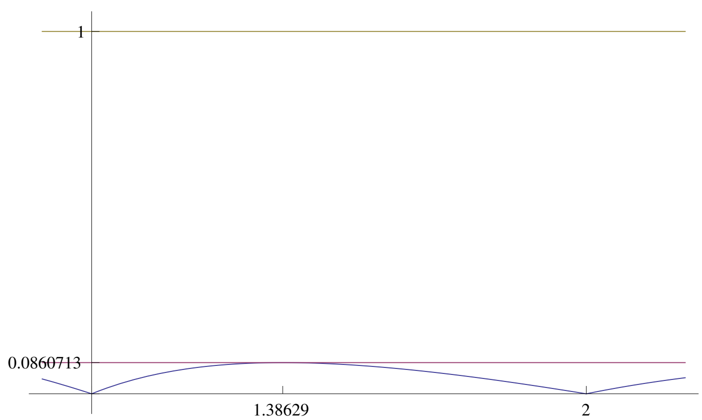
In[3]:=
```
θ[x_] := ⌈x⌉ - x
```

In[4]:=
```
ϵ[x_] := β[θ[Log2[x]]]
```

Function ϵ oscillates between 0 and
$1 - \lg e + \lg \lg e \approx 0.08607133205593432$

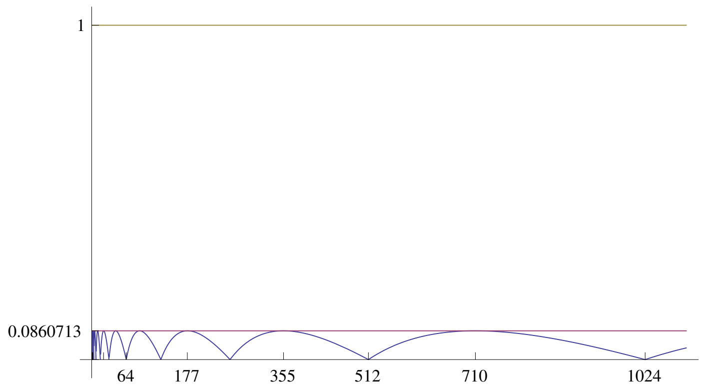
In[33]:=
```
Plot[{ϵ[n], 0.0860713, 1}, {n, 0.9, 2.2}, PlotTheme → "Classic",
 PlotStyle → {Thickness[Small]}, AspectRatio → 0.6,
 Ticks → {{1, 1.38629, 2, 22, 64, 177, 355, 512, 710, 1024}, {0, 0.0860713, 1}}]
```

Out[33]=



In[32]:=
```
Plot[{ϵ[n], 0.08607133205593431`, 1}, {n, 1, 1100},
 PlotTheme → "Classic", PlotStyle → {Thickness[Small]}, AspectRatio → .6,
 Ticks → {{1, 2^.4712336270551024`, 2, Round[2^4.4712336270551024`],
    64, Round[2^7.4712336270551024`], Round[2^8.4712336270551024`], 512,
    Round[2^9.4712336270551024`], 1024}, {0, 0.08607133205593431`, 1}}]
```

Out[32]=

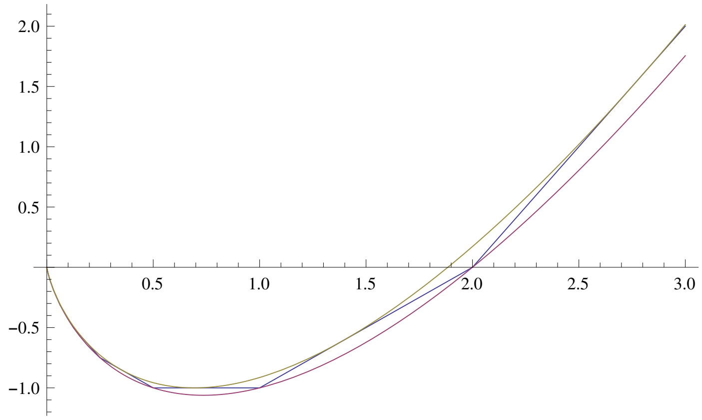


First I will prove that

$x \lceil Log2[x] \rceil - 2^{\lceil Log2[x] \rceil} = x\,(Log2[x] + \epsilon[x] - 1)$

In[34]:=
```
Plot[Tooltip[{x ⌈Log2[x]⌉ - 2^⌈Log2[x]⌉, x (Log2[x] + 0 - 1), x (Log2[x] + 0.086 - 1)}],
 {x, 0.00001, 3}, PlotTheme → "Classic", PlotStyle → {Thickness[Small]}]
```

Out[34]=



By the definition of ϵ we have

$\epsilon[x] = 1 + \lceil Log2[x] \rceil - Log2[x] - 2^{\lceil Log2[x] \rceil - Log2[x]}$

Hence

$$\epsilon[x] = 1 + \lceil Log2[x] \rceil - Log2[x] - \frac{2^{\lceil Log2[x] \rceil}}{2^{Log2[x]}}$$

$$\epsilon[x] = 1 + \lceil Log2[x] \rceil - Log2[x] - \frac{2^{\lceil Log2[x] \rceil}}{x}$$

$$Log2[x] + \epsilon[x] - 1 = \lceil Log2[x] \rceil - \frac{2^{\lceil Log2[x] \rceil}}{x}$$

$x\,(Log2[x] + \epsilon[x] - 1) = x \lceil Log2[x] \rceil - 2^{\lceil Log2[x] \rceil}$

This completes the proof of
$x \lceil Log2[x] \rceil - 2^{\lceil Log2[x] \rceil} = x\,(Log2[x] + \epsilon[x] - 1)$

Second, I will prove that

$x \lfloor Log2[x] \rfloor - 2^{\lfloor Log2[x] \rfloor + 1} = x \lceil Log2[x] \rceil - 2^{\lceil Log2[x] \rceil} - x$

1) For $Log2[x] \in \mathbb{Z}$, $\lceil Log2[x] \rceil = Log2[x] = \lfloor Log2[x] \rfloor$

So,

$x \lfloor Log2[x] \rfloor - 2^{\lfloor Log2[x] \rfloor + 1} = x\,Log2[x] - 2^{Log2[x]+1} = x\,Log2[x] - 2 \times 2^{Log2[x]} =$
$x\,Log2[x] - 2\,x = x\,Log2[x] - x - x = x\,Log2[x] - 2^{Log2[x]} - x = x \lceil Log2[x] \rceil - 2^{\lceil Log2[x] \rceil} - x$

2) For $x \notin \mathbb{Z}$, $\lceil x \rceil - \lfloor x \rfloor = 1$, so

$\lfloor x \rfloor = \lceil x \rceil - 1.$

Thus for $Log2[x] \notin \mathbb{Z}$

$\lfloor Log2[x] \rfloor = \lceil Log2[x] \rceil - 1$

So

$x \lfloor Log2[x] \rfloor - 2^{\lfloor Log2[x] \rfloor + 1} =$
$x\,(\lceil Log2[x] \rceil - 1) - 2^{\lceil Log2[x] \rceil - 1 + 1} = x\,(\lceil Log2[x] \rceil - 1) - 2^{\lceil Log2[x] \rceil - 1 + 1} = x\,(\lceil Log2[x] \rceil) - 2^{\lceil Log2[x] \rceil} - x$

This completes the proof of

$x \lfloor \text{Log2}[x] \rfloor - 2^{\lfloor \text{Log2}[x] \rfloor + 1} = x \lceil \text{Log2}[x] \rceil - 2^{\lceil \text{Log2}[x] \rceil} - x$

Third, I will prove that

$x \lfloor \text{Log2}[x] \rfloor - 2^{\lfloor \text{Log2}[x] \rfloor + 1} = x\,(\text{Log2}[x] + \epsilon[x] - 2)$

Combining

$x \lceil \text{Log2}[x] \rceil - 2^{\lceil \text{Log2}[x] \rceil} = x\,(\text{Log2}[x] + \epsilon[x] - 1)$

with

$x \lfloor \text{Log2}[x] \rfloor - 2^{\lfloor \text{Log2}[x] \rfloor + 1} = x \lceil \text{Log2}[x] \rceil - 2^{\lceil \text{Log2}[x] \rceil} - x$

yields

$x \lfloor \text{Log2}[x] \rfloor - 2^{\lfloor \text{Log2}[x] \rfloor + 1} = x\,(\text{Log2}[x] + \epsilon[x] - 1) - x =$

$x\,(\text{Log2}[x] + \epsilon[x] - 2)$

This completes the proof of

$x \lfloor \text{Log2}[x] \rfloor - 2^{\lfloor \text{Log2}[x] \rfloor + 1} = x\,(\text{Log2}[x] + \epsilon[x] - 2)$

Now, putting $x = n + 1$ we get

$$(n + 1)\,\lfloor \text{Log2}[n + 1] \rfloor - 2^{\lfloor \text{Log2}[n+1] \rfloor + 1} = (n + 1)\,(\text{Log2}[n + 1] + \epsilon[n + 1] - 2)$$

This completes the proof.

Optional interesting fact

The function

$\beta[x] = 1 + \text{Log2}[x] - \lfloor \text{Log2}[x] \rfloor - 2^{\text{Log2}[x] - \lfloor \text{Log2}[x] \rfloor}$

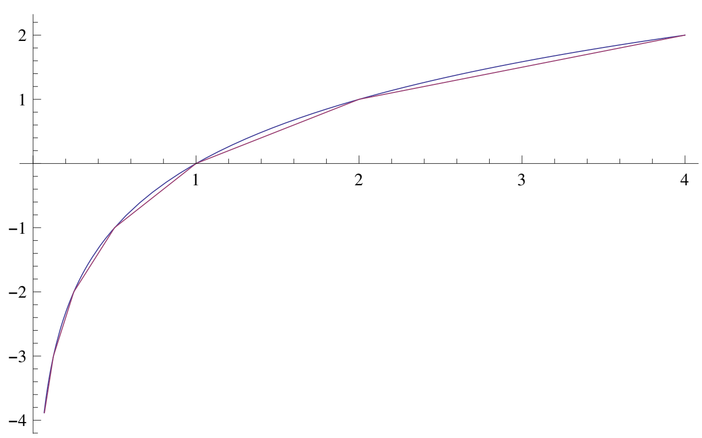
similar to ϵ

may be used to linearly interpolate $\lg_2$

In[35]:=

```
Plot[Tooltip[{Log2[x], Log2[x] - (1 + Log2[x] - ⌊Log2[x]⌋ - 2^(Log2[x] - ⌊Log2[x]⌋))}],
 {x, 0.00001, 4}, PlotTheme → "Classic", PlotStyle → {Thickness[Small]}]
```

Out[35]=



In[36]:=
```
Plot[Tooltip[{Log2[x], Log2[x] - 0.08607133205593431`,
   Log2[x] - (1 + Log2[x] - ⌊Log2[x]⌋ - 2^(Log2[x] - ⌊Log2[x]⌋))}],
 {x, 0.00001, 4}, PlotTheme → "Classic", PlotStyle → {Thickness[Small]}]
```

Out[36]=