$\sum_{i=1}^{n} i$

$\frac{1}{2} n (1 + n)$

$\sum_{i=1}^{n} i^2$

$\frac{1}{6} n (1 + n) (1 + 2 n)$

$\sum_{i=1}^{n} i^3$

$\frac{1}{4} n^2 (1 + n)^2$

$\sum_{i=1}^{n} i^4$

$\frac{1}{30} n (1 + n) (1 + 2 n) \left(-1 + 3 n + 3 n^2\right)$

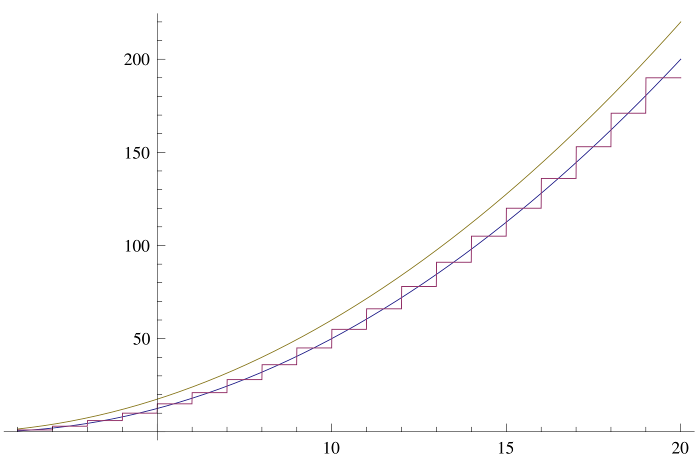
Expand[%]

$-\frac{n}{30} + \frac{n^3}{3} + \frac{n^4}{2} + \frac{n^5}{5}$

$\int_0^n x \, dx \leq \sum_{i=1}^{n} i \leq \int_1^{n+1} x \, dx$

$\frac{n^2}{2} \leq \frac{1}{2} n (1 + n) \leq n + \frac{n^2}{2}$

Plot$\left[\left\{\int_0^n x \, dx, \sum_{i=1}^{n} i, \int_1^{n+1} x \, dx\right\}, \{n, 1, 20\}\right]$



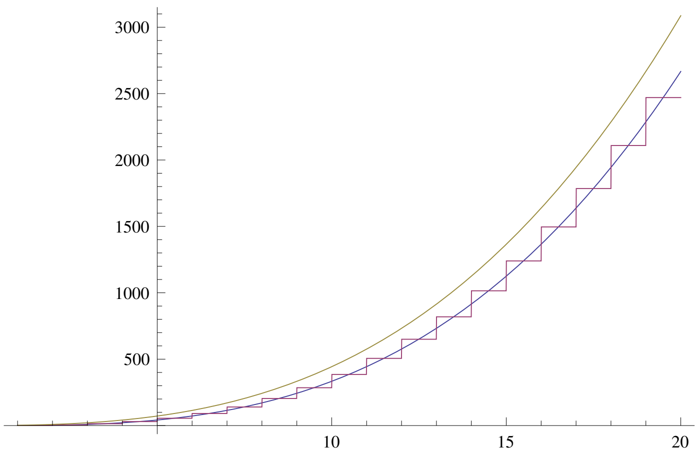
$\int_0^n x^2 \, dx \leq \sum_{i=1}^{n} i^2 \leq \int_1^{n+1} x^2 \, dx$

$\frac{n^3}{3} \leq \frac{1}{6} n (1 + n) (1 + 2 n) \leq n + n^2 + \frac{n^3}{3}$

Expand[%]

$$\frac{n^3}{3} \leq \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} \leq n + n^2 + \frac{n^3}{3}$$

$$\mathbf{Plot}\left[\left\{\int_0^n x^2\,dx,\ \sum_{i=1}^n i^2,\ \int_1^{n+1} x^2\,dx\right\},\ \{n, 1, 20\}\right]$$



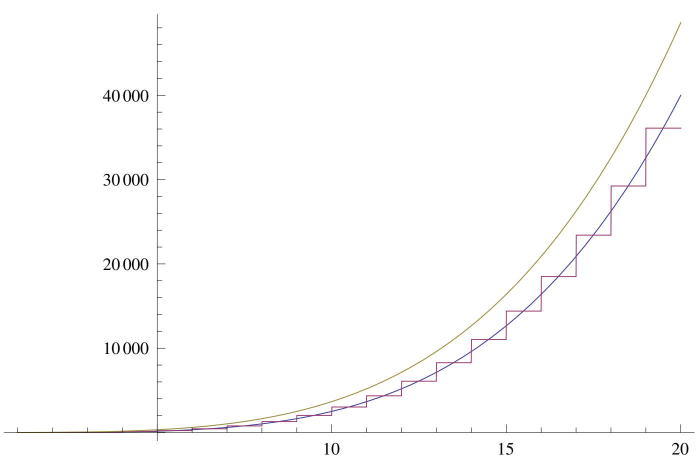
$$\int_0^n x^3\,dx \leq \sum_{i=1}^n i^3 \leq \int_1^{n+1} x^3\,dx$$

$$\frac{n^4}{4} \leq \frac{1}{4} n^2 (1+n)^2 \leq \frac{1}{4}\left(-1 + (1+n)^4\right)$$

Expand[%]

$$\frac{n^4}{4} \leq \frac{n^2}{4} + \frac{n^3}{2} + \frac{n^4}{4} \leq n + \frac{3 n^2}{2} + n^3 + \frac{n^4}{4}$$

$$\mathbf{Plot}\left[\left\{\int_0^n x^3\,dx,\ \sum_{i=1}^n i^3,\ \int_1^{n+1} x^3\,dx\right\},\ \{n, 1, 20\}\right]$$



$$\int_0^n x^k\,dx \leq \sum_{i=1}^n i^k \leq \int_1^{n+1} x^k\,dx$$

$$\text{ConditionalExpression}\left[\frac{n^{1+k}}{1+k} \leq \text{HarmonicNumber}[n, -k] \leq \left(-1 + (1+n)^{1+k}\right) / (1+k),\ \text{Re}[k] > -1\ \&\&\ (\text{Re}[n] \geq -1\ ||\ n \notin \text{Reals})\right]$$

$$\frac{n^{1+k}}{1+k} \le \sum_{i=1}^{n} i^k \le \left(-1 + (1+n)^{1+k}\right) / (1+k)$$

$$\frac{n^{k+1}}{k+1} \le \sum_{i=1}^{n} i^k \le \left((n+1)^{k+1} - 1\right) / (k+1)$$

Theorem - part one

For every non-decreasing function f

$$\int_{a-1}^{b} f(x)\, dx \le \sum_{i=a}^{b} f(i) \le \int_{a}^{b+1} f(x)\, dx$$

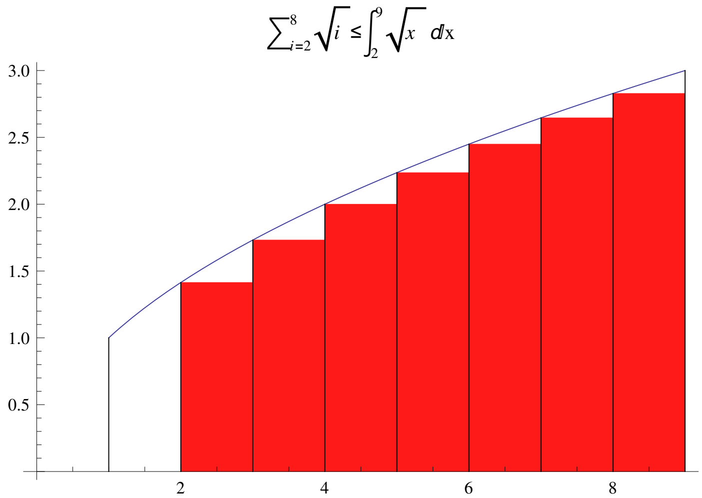
Example

$$\int_{1}^{8} \sqrt{x}\, dx \le \sum_{i=2}^{8} \sqrt{i} \le \int_{2}^{9} \sqrt{x}\, dx$$

True

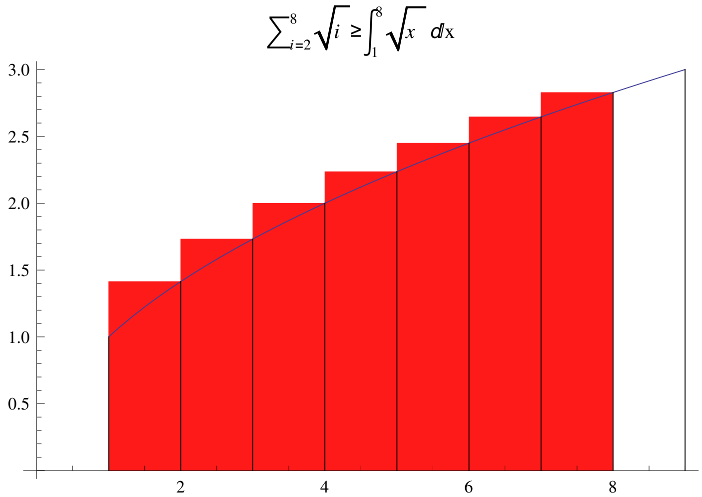
Graphic proof of the second inequality

$$\sum_{i=2}^{8} \sqrt{i} \le \int_{2}^{9} \sqrt{x}\, dx :$$



Graphic proof of the first inequality

$\int_1^8 \sqrt{x}\, dx \le \sum_{i=2}^{8} \sqrt{i}$:



$\int_{1.50574838034444}^{8.50574838034444} \sqrt{x}\, dx \approx \sum_{i=2}^{8} \sqrt{i}$

$\approx 15.30600052603572$

Theorem - part two

For every non - increasing function f

$\int_a^{b+1} f(x)\, dx \le \sum_{i=a}^{b} f(i) \le \int_{a-1}^{b} f(x)\, dx$

Anther usefull Theorem

For every non - decreasing function f and every

$1 \le k \le n$,

$k \times f(n-k+1) \le \sum_{i=1}^{n} f(i) \le n \times f(n)$

or, substituting n - k + 1 for k,

$(n-k+1) \times f(k) \le \sum_{i=1}^{n} f(i) \le n \times f(n)$

Example

$k \times (n-k+1)! \le \sum_{i=1}^{n} i! \le n \times n!$

Putting $k = \left\lceil \frac{n}{2} \right\rceil$ yields :

$\left\lceil \frac{n}{2} \right\rceil \times \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right)! \le \sum_{i=1}^{n} i! \le n \times n!$

By $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = n$ we get :

$\left\lceil \frac{n}{2} \right\rceil \times \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)! \le \sum_{i=1}^{n} i! \le n \times n!$

By $\left\lceil \frac{n}{2} \right\rceil \le \left\lfloor \frac{n}{2} \right\rfloor + 1 = n$ we get :

$$\left\lceil \frac{n}{2} \right\rceil \times \left\lceil \frac{n}{2} \right\rceil ! \le \sum_{i=1}^{n} i\,! \le n \times n\,!$$

By Stirling formula we get :

$$\left\lceil \frac{n}{2} \right\rceil \left(\frac{\left\lceil \frac{n}{2} \right\rceil}{e} \right)^{\left\lceil \frac{n}{2} \right\rceil} \sqrt{2 \pi \left\lceil \frac{n}{2} \right\rceil} < \sum_{i=1}^{n} i\,! < n \left(\frac{n}{e} \right)^{n} \sqrt{2 \pi n} \left(1 + \frac{1}{11 n} \right)$$

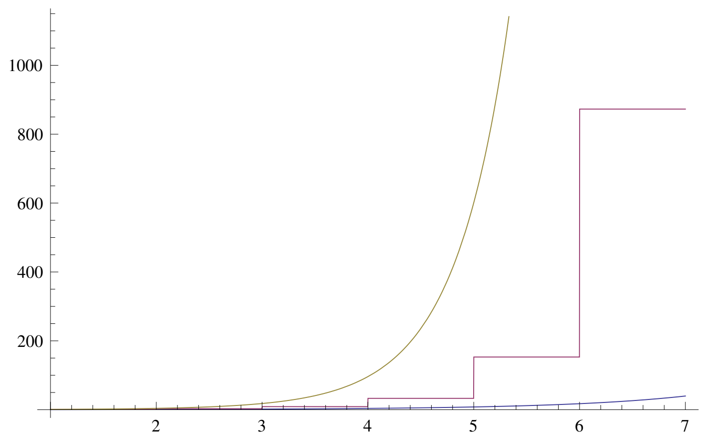
By $\frac{n}{2} \le \left\lceil \frac{n}{2} \right\rceil$ we get :

$$\frac{n}{2} \left(\frac{\frac{n}{2}}{e} \right)^{\frac{n}{2}} \sqrt{2 \pi \frac{n}{2}} < \sum_{i=1}^{n} i\,! < n \left(\frac{n}{e} \right)^{n} \sqrt{2 \pi n} \left(1 + \frac{1}{11 n} \right)$$

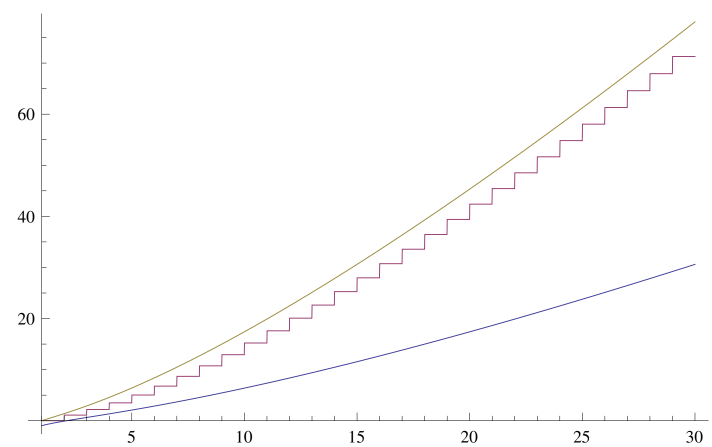
or

$$e \left(\frac{n}{2 e} \right)^{\frac{n}{2}+1} \sqrt{\pi n} < \sum_{i=1}^{n} i\,! < e \left(\frac{n}{e} \right)^{n+1} \sqrt{2 \pi n} \left(1 + \frac{1}{11 n} \right)$$

$$\text{Plot}\left[\left\{ e \left(\frac{n}{2 e} \right)^{\frac{n}{2}+1} \sqrt{\pi n},\ \sum_{i=1}^{n} i\,!,\ e \left(\frac{n}{e} \right)^{n+1} \sqrt{2 \pi n} \left(1 + \frac{1}{11 n} \right) \right\}, \{n, 1, 7\}\right]$$



$$\text{Plot}\left[\left\{ \text{Log}\left[e \left(\frac{n}{2 e} \right)^{\frac{n}{2}+1} \sqrt{\pi n} \right],\ \text{Log}\left[\sum_{i=1}^{n} i\,! \right],\ \text{Log}\left[e \left(\frac{n}{e} \right)^{n+1} \sqrt{2 \pi n} \left(1 + \frac{1}{11 n} \right) \right] \right\}, \{n, 1, 30\}\right]$$



If non - decreasing f is non - negative then for every
$1 \le k \le n$,

$$f(n-k+1)^k \le \prod_{i=1}^{n} f(i) \le f(n)^n$$

Example

$$(n-k+1)^k \le \prod_{i=1}^{n} i \le n^n$$

where $\prod_{i=1}^{n} i = n!$

So,

$$(n-k+1)^k \le n! \le n^n$$

If n is even then putting $k = \frac{n}{2}$ yields :

$$\left(\frac{n}{2}+1\right)^{\frac{n}{2}} \le n! \le n^n$$

Putting $k = \left\lceil \frac{n}{2} \right\rceil$ in $(n-k+1)^k \le n! \le n^n$

we get

$$\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right)^{\left\lceil \frac{n}{2} \right\rceil} \le n! \le n^n$$

or, by $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = n$,

$$\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)^{\left\lceil \frac{n}{2} \right\rceil} \le n! \le n^n$$

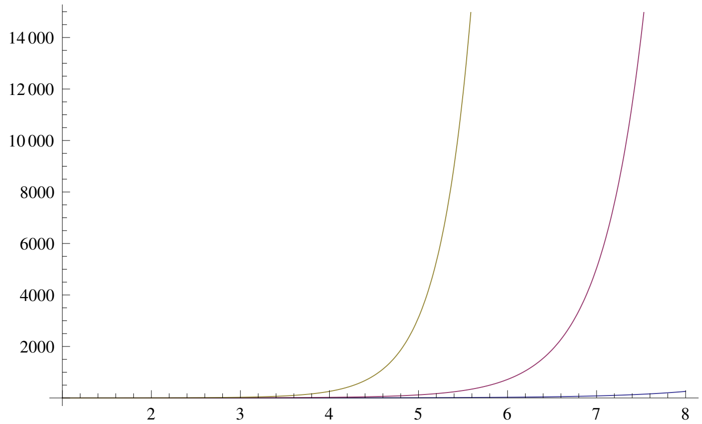
By $\left\lceil \frac{n}{2} \right\rceil < \left\lfloor \frac{n}{2} \right\rfloor + 1$ we get :

$$\left(\left\lceil \frac{n}{2} \right\rceil\right)^{\left\lceil \frac{n}{2} \right\rceil} \le n! \le n^n$$

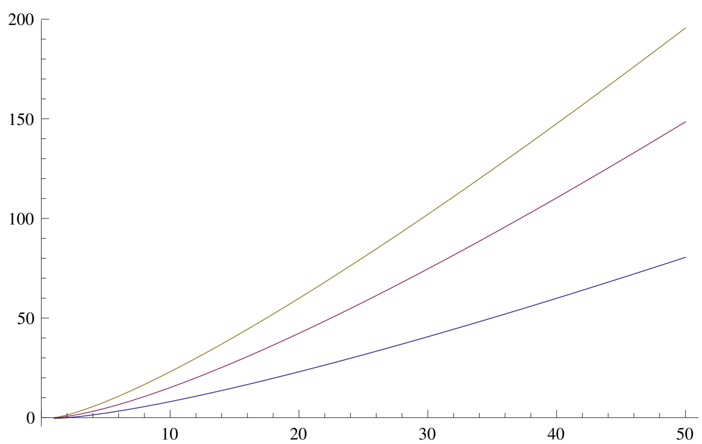
and by $\frac{n}{2} \le \left\lceil \frac{n}{2} \right\rceil$ we conclude

$$\left(\frac{n}{2}\right)^{\frac{n}{2}} \le n! \le n^n$$

Plot$\left[\left\{\left(\frac{n}{2}\right)^{\frac{n}{2}}, n!, n^n\right\}, \{n, 1, 8\}\right]$



Plot$\left[\left\{\text{Log}\left[\left(\frac{n}{2}\right)^{\frac{n}{2}}\right], \text{Log}[n!], \text{Log}[n^n]\right\}, \{n, 1, 50\}\right]$



** ** ** ** ** ** ** ** **

Undergraduate students : skip until ##

Proof of $\sum$ part

There are m - n elements anfter m in an n - element sequence. So,
since f is non - decreasing, in the sequence
f (1), f (2), ..., f (m), ..., f (n)
there are at least n - m + 1
elements greater of equal to f (m).

Therefore,
$(n - m + 1) \times f(m) \leq \sum_{i=1}^{n} f(i)$

Puting k = n - m + 1, that is, m = n - k + 1, yields

$k \times f(n-k+1) \leq \sum_{i=1}^{n} f(i).$

The inequality

$$\sum_{i=1}^{n} f(i) \leq n \times f(n)$$

is easy to proof and is left as an exercise.

Proof of the $\prod$ part from the $\sum$ part

$$k \times \log_2 f(n-k+1) \leq \sum_{i=1}^{n} \log_2 f(i) \leq n \times \log_2 f(n)$$

$$\log_2 f(n-k+1)^k \leq \log_2 \prod_{i=1}^{n} f(i) \leq \log_2 f(n)^n$$

$$f(n-k+1)^k \leq \prod_{i=1}^{n} f(i) \leq f(n)^n$$

##

$$\sum_{i=0}^{k} 2^i$$

$$-1 + 2^{1+k}$$

$$\sum_{i=0}^{k} r^i$$

$$\frac{-1 + r^{1+k}}{-1 + r}$$

$$\sum_{i=0}^{k} \frac{1}{2^i}$$

$$2^{-k} \left(-1 + 2^{1+k}\right)$$

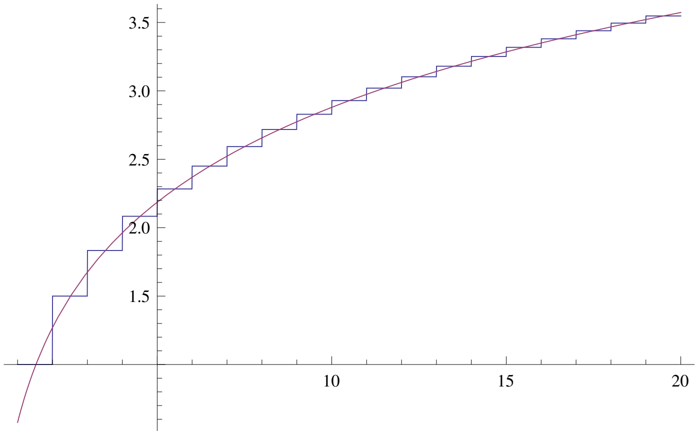
$$\sum_{i=1}^{n} \frac{1}{i}$$

HarmonicNumber[n]

Euler ' s approximation of harmonic number

$$\sum_{i=1}^{n} \frac{1}{i} \approx \mathrm{Log}[n] + .577$$

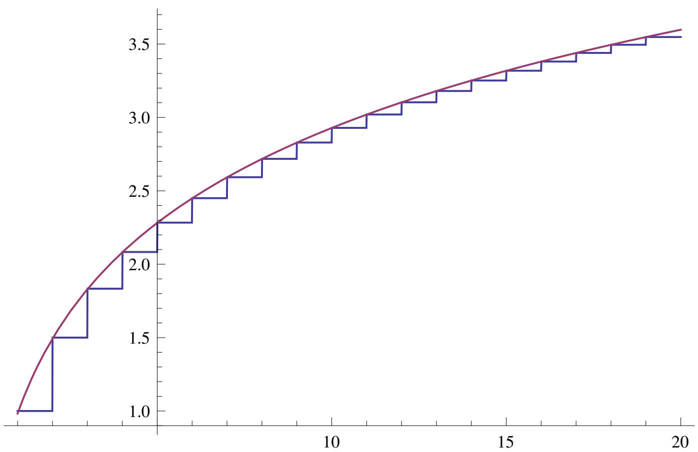
```
Plot[Tooltip[{∑_{i=1}^n 1/i, Log[n] + .577}], {n, 1, 20}]
```



A glance at the above graph yields the following improvement

In[19]:=
```
Plot[Tooltip[{∑_{i=1}^n 1/i, Log[n + 0.5] + 0.577}], {n, 1, 20}, PlotTheme → "Classic"]
```

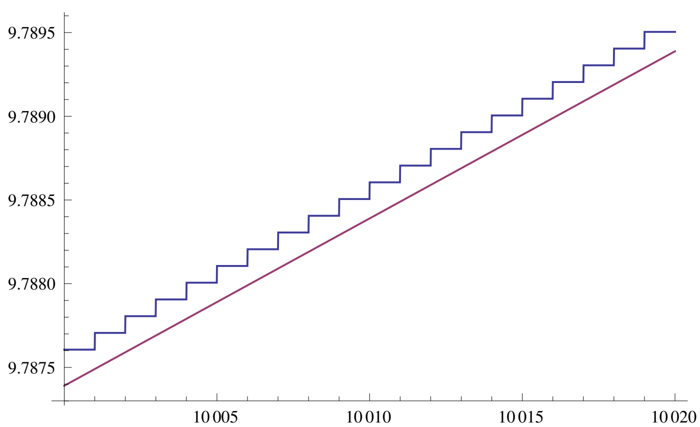
Out[19]=



But it ' s not so good for larger n

In[20]:=
```
Plot[Tooltip[{∑_{i=1}^n 1/i, Log[n + 0.5] + 0.577}], {n, 10 000, 10 020}, PlotTheme → "Classic"]
```

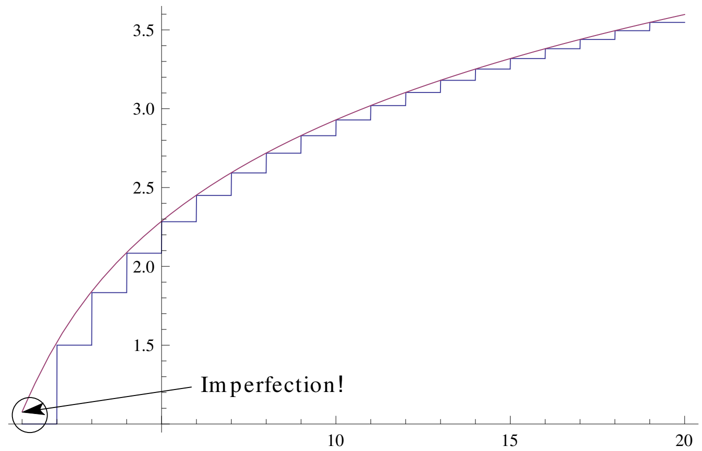
Out[20]=



A more precise traditional approximation

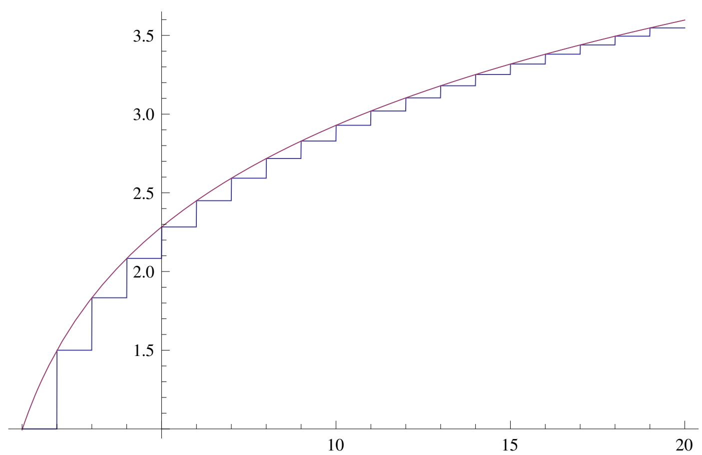
```
Plot[Tooltip[{Sum[1/i, {i, 1, n}], Log[n] + .577 + 1/(2 n)}], {n, 1, 20}]
```

$$\text{Plot}\left[\text{Tooltip}\left[\left\{\sum_{i=1}^{n}\frac{1}{i},\ \text{Log}[n]+.577+\frac{1}{2\,n}\right\}\right],\ \{n,\ 1,\ 20\}\right]$$



Ever better approximation, if you are a perfectionist :

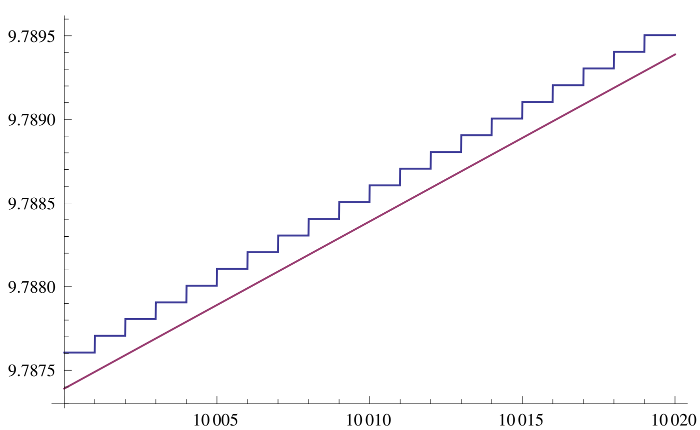
$$\text{Plot}\left[\text{Tooltip}\left[\left\{\sum_{i=1}^{n}\frac{1}{i},\ \text{Log}[n]+.577+\frac{1}{2\,n}-\frac{1}{12\,n^2}\right\}\right],\ \{n,\ 1,\ 20\}\right]$$



Still not so good for larger n

In[18]:= $$\text{Plot}\left[\text{Tooltip}\left[\left\{\sum_{i=1}^{n}\frac{1}{i},\ \text{Log}[n]+0.577+\frac{1}{2\,n}-\frac{1}{12\,n^2}\right\}\right],\ \{n,\ 10\,000,\ 10\,020\},\ \text{PlotTheme}\rightarrow\text{"Classic"}\right]$$

Out[18]=



You can use the constant EulerGamma =

N[EulerGamma]

0.577216

0.5772156649015329`

in lieu of .577

In[22]:= $N\left[e^{\text{EulerGamma}}\right]$

Out[22]= 1.78107

End of Euler's approximation of harmonic number

Here is another (less accurate) way of approximating

the harmonic series $\sum_{i=1}^{n} \frac{1}{i}$. It's a good exercise.

$$\int_2^{n+1} \frac{1}{x}\,dx \le \sum_{i=2}^{n} \frac{1}{i} \le \int_1^{n} \frac{1}{x}\,dx$$

ConditionalExpression[
 -Log[2] + Log[1 + n] ≤ -1 + HarmonicNumber[n] ≤ Log[n], Re[n] ≥ 0 || n ∉ Reals]

$$-\text{Log}[2] + \text{Log}[1+n] \le \sum_{i=2}^{n} \frac{1}{i} \le \text{Log}[n]$$

N[Log[2]]

0.693147

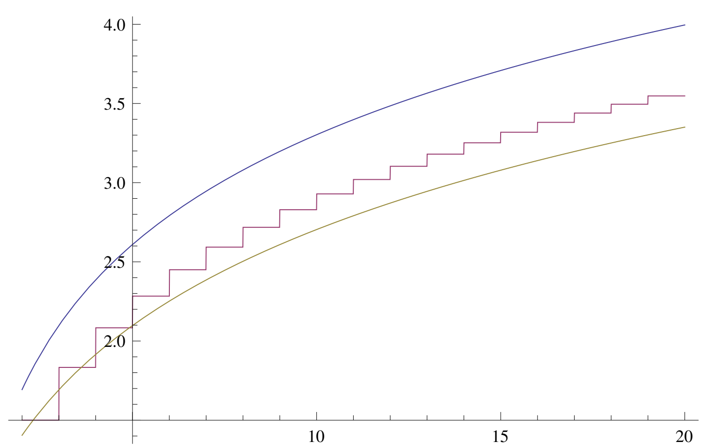
So,

$$\text{Log}[n+1] - 0.694 \le \sum_{i=2}^{n} \frac{1}{i} \le \text{Log}[n]$$

or

$$\text{Log}[n+1] + 0.306 \le \sum_{i=1}^{n} \frac{1}{i} \le \text{Log}[n] + 1$$

$$\text{Plot}\left[\text{Tooltip}\left[\left\{\text{Log}[n]+1, \sum_{i=1}^{n} \frac{1}{i}, \text{Log}[n+1]+0.306\right\}\right], \{n, 2, 20\}\right]$$



$\sum_{i=1}^{k} i \times 2^i$

$2 \left(1 - 2^k + 2^k k\right)$

Expand[%]

$2 - 2^{1+k} + 2^{1+k} k$

$2 + 2^{1+k} (k - 1)$

$2^{k+1} (k - 1) + 2$

Log2[n !] = $\sum_{i=1}^{n}$ Log2[i]

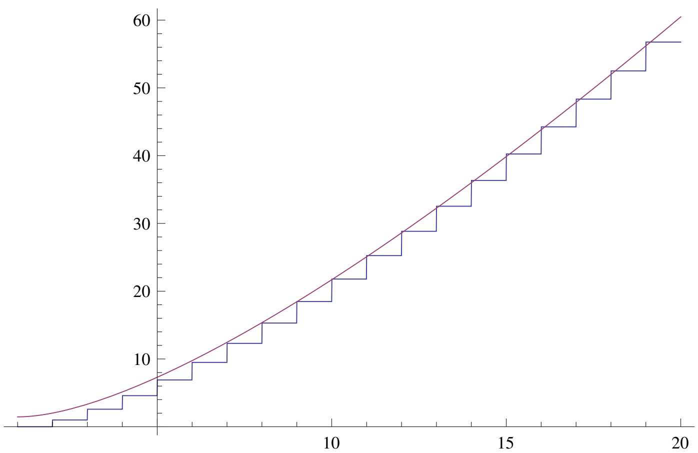
$\sum_{i=1}^{n}$ Log2[i]

Log[Pochhammer[1, n]] / Log[2]

Textbook ' s approximation :

$\sum_{i=1}^{n}$ Log2[i] ≈ n Log2[n] - 1.443 n + 2.9

Plot[Tooltip[{$\sum_{i=1}^{n}$ Log2[i], n Log2[n] - 1.443 n + 2.9}], {n, 1, 20}]



Even more accurate approximation is in the Mathematica / Log_of _factorial.nb :

$\sum_{i=1}^{n}$ Log2[i] ≈

n Log2[n] - $\frac{1}{\text{Log}[2]}$ n + $\frac{1}{2}$Log2[n] + Log[2 π] / Log[4] + 1 / (Log[4096] n) - 1 / (360 Log[2] n^3)

This yields :

$\sum_{i=1}^{n}$ Log2[i] ≈

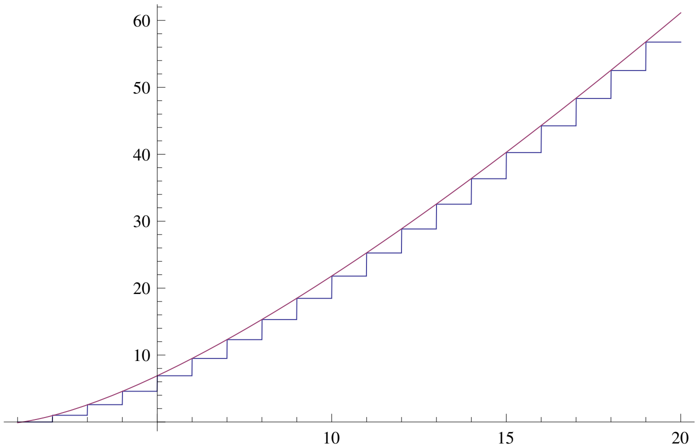
$\left(n + \frac{1}{2}\right)$ Log2[n] - 1.4426950408889634` n + 1.3257480647361592`

or

$\sum_{i=1}^{n}$ Log2[i] ≈

$\left(n + \frac{1}{2}\right)$ Log2[n] - 1.443 n + 1.326

$\text{Plot}\left[\text{Tooltip}\left[\left\{\sum_{i=1}^{n} \text{Log2}[i],\ \left(n + \frac{1}{2}\right) \text{Log2}[n] - 1.44\, n + 1.33\right\}\right],\ \{n,\ 1,\ 20\}\right]$



Sterling formula (see file Mathematica / Stirling_formula.nb) would yield this :

$$\sum_{i=1}^{n} \text{Log2}[i] = \text{Log2}[n!] > \text{Log2}\left[\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right] > \left(n + \frac{1}{2}\right) \text{Log2}[n] - \frac{n}{\text{Log}[2]} + \frac{1}{2} \text{Log2}[2\pi] \approx$$

$$\left(n + \frac{1}{2}\right) \text{Log2}[n] - 1.4426950408889634\grave{}\ n + 1.3257480647361592\grave{} \approx$$

$$\left(n + \frac{1}{2}\right) \text{Log2}[n] - 1.443\, n + 1.325$$

and

$$\sum_{i=1}^{n} \text{Log2}[i] = \text{Log2}[n!] < \text{Log2}\left[\left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(1 + \frac{1}{11\, n}\right)\right] <$$

$$\left(n + \frac{1}{2}\right) \text{Log2}[n] - \frac{n}{\text{Log}[2]} + \frac{1}{2} \text{Log2}[2\pi] + \text{Log2}\left[1 + \frac{1}{11\, n}\right].$$

Since Log2[x] is a convex function of x

$\left(\text{its derivative, } \frac{1}{x} \text{Log2}[e] \text{ is a decreasing function of } x\right),$

$$\text{Log2}[x + y] < \text{Log2}[x] + \frac{\text{Log2}[e]}{x} (y - x).$$

Hence,

$$\text{Log2}\left[1 + \frac{1}{11\, n}\right] < \text{Log2}[1] + \frac{\text{Log2}[e]}{1} \left(1 + \frac{1}{11\, n} - 1\right) =$$

$$= \frac{\text{Log2}[e]}{11\, n}.$$

$$N\left[\frac{\text{Log2}[e]}{11\, n}\right]$$

$$\frac{0.131154}{n}$$

0.1311540946262694` / n

So,

$$\left(n+\frac{1}{2}\right)\text{Log2}[n]-1.4426950408889634\grave{}\,n+1.3257480647361592\grave{}+\text{Log2}\left[1+\frac{1}{11\,n}\right]<$$

$$\left(n+\frac{1}{2}\right)\text{Log2}[n]-1.4426950408889634\grave{}\,n+1.3257480647361592\grave{}+\frac{\text{Log2}[e]}{11\,n}=$$

$$=\left(n+\frac{1}{2}\right)\text{Log2}[n]-1.4426950408889634\grave{}\,n+1.3257480647361592\grave{}+0.1311540946262694\grave{}/n<$$

$$<\left(n+\frac{1}{2}\right)\text{Log2}[n]-1.442\,n+1.326+\frac{0.132}{n}$$

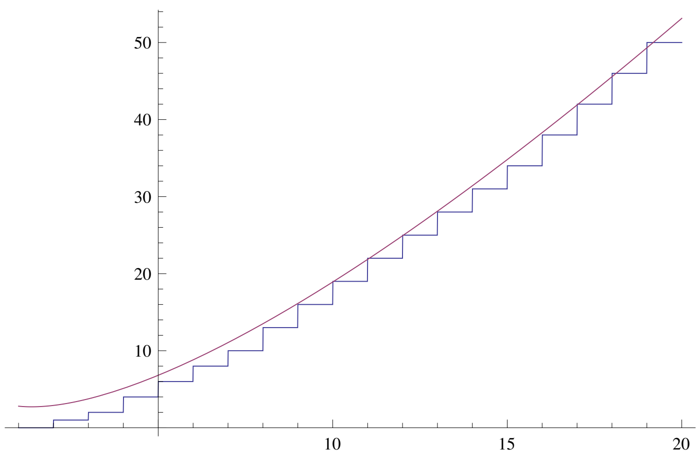
More tricky sums

$$\sum_{i=1}^{n}\lfloor \text{Log2}[i]\rfloor$$

Important approximation :

$$\sum_{i=1}^{n}\lfloor \text{Log2}[i]\rfloor \approx n\,\text{Log2}[n]-1.9\,n+4.7$$

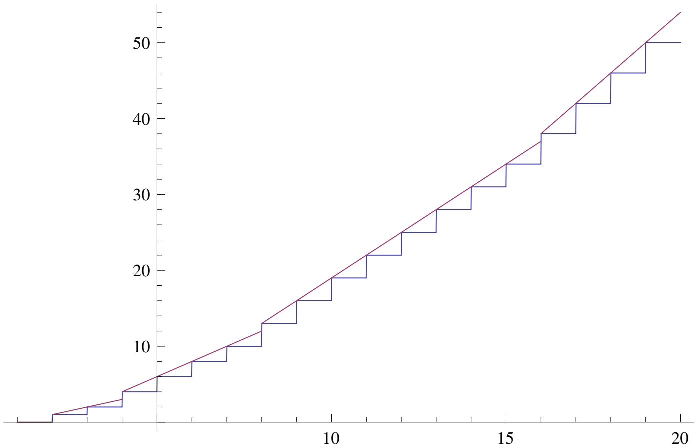
```
Plot[Tooltip[{∑_{i=1}^{n} ⌊Log2[i]⌋, n Log2[n] - 1.9 n + 4.7}], {n, 1, 20}]
```



The exact value :

$$\sum_{i=1}^{n}\lfloor \text{Log2}[i]\rfloor = (n+1)\,\lfloor \text{Log2}[n]\rfloor - 2^{\lfloor \text{Log2}[n]\rfloor+1}+2$$

```
Plot[Tooltip[{∑_{i=1}^{n} ⌊Log2[i]⌋, (n + 1) ⌊Log2[n]⌋ - 2^(⌊Log2[n]⌋+1) + 2}], {n, 1, 20}]
```



Also,

$$\sum_{i=1}^{n} \lfloor \text{Log2}[i] \rfloor = (n+1)\lfloor \text{Log2}[n+1] \rfloor - 2^{\lfloor \text{Log2}[n+1] \rfloor + 1} + 2 =$$

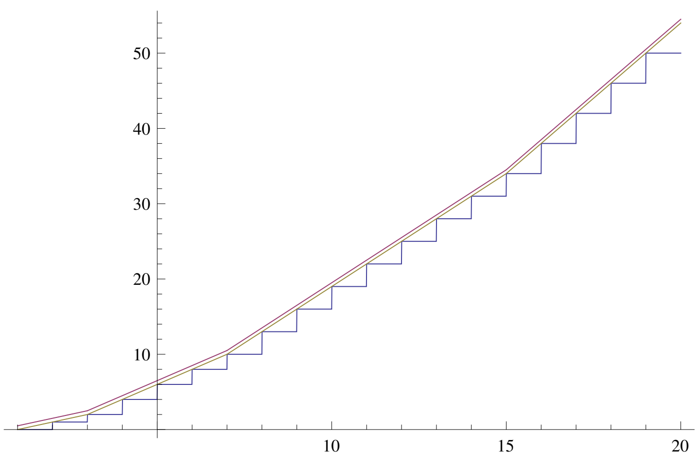
[The above is Knuth ' s formula]

$$= (n+1)\,(\text{Log2}[n+1] + \epsilon[n+1]) - 2n$$

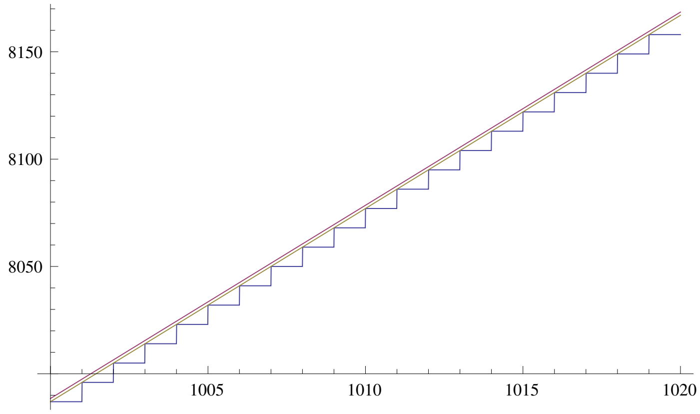
where

```
β[x_] := 1 + x - 2^x

θ[x_] := ⌈x⌉ - x

ϵ[x_] := β[θ[Log2[x]]]
```

```
Plot[Tooltip[{∑_{i=1}^{n} ⌊Log2[i]⌋, (n + 1) ⌊Log2[n + 1]⌋ - 2^(⌊Log2[n+1]⌋+1) + 2 + .5 (* + .5 added
    to see separate lines*), (n + 1) (Log2[n + 1] + ϵ[n + 1]) - 2 n}], {n, 1, 20}]
```

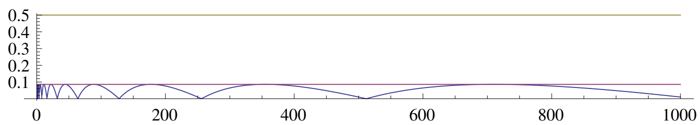


```
Plot[Tooltip[{∑_{i=1}^n ⌊Log2[i]⌋, (n + 1) ⌊Log2[n + 1]⌋ - 2^(⌊Log2[n+1]⌋+1) + 2 +1.5 (* + 1.5 added to
    see separate lines*), (n + 1) (Log2[n + 1] + ϵ[n + 1]) - 2 n}], {n, 1000, 1020}]
```



Here is a plot of function ϵ[n]

```
Plot[{ϵ[n], .0860, .5}, {n, 1, 1000}, AspectRatio → .13]
```

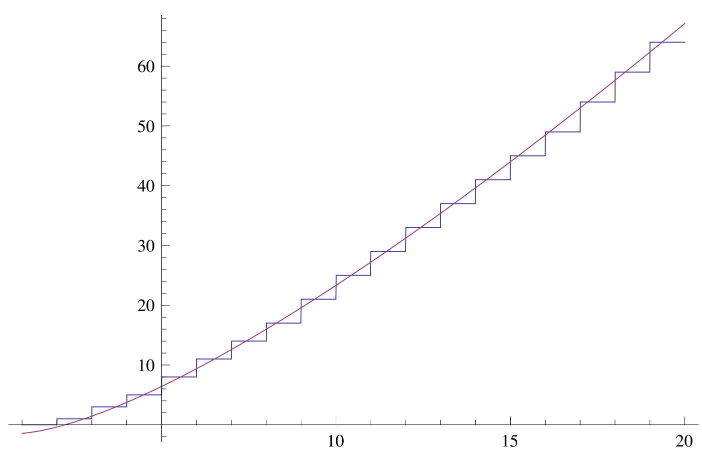


The rest of this file is optional

Approximation of sums of ceilings of logs :

$$\sum_{i=1}^{n} \lceil Log2[i] \rceil \approx n\, Log2[n] - 0.94\, n - 0.5$$

```
Plot[Tooltip[{∑_{i=1}^n ⌈Log2[i]⌉, n Log2[n] - 0.94 n - 0.5}], {n, 1, 20}]
```



The exact value :

$$\sum_{i=1}^{n} \lceil \text{Log2}[i] \rceil = \sum_{i=2}^{n} \lceil \text{Log2}[i] \rceil = (* \text{ by } \lfloor \log_2 k \rfloor + 1 = \lceil \log_2 (k+1) \rceil \; *) \sum_{i=2}^{n} (\lfloor \text{Log2}[i-1] \rfloor + 1) =$$
$$\sum_{i=1}^{n-1} \lfloor \text{Log2}[i] \rfloor + n - 1 = n \lfloor \text{Log2}[n-1] \rfloor - 2^{\lfloor \text{Log2}[n-1] \rfloor + 1} + 2 + n - 1 =$$
$$n (\lfloor \text{Log2}[n-1] \rfloor + 1) - 2^{\lfloor \text{Log2}[n-1] \rfloor + 1} + 1 = n (\lceil \text{Log2}[n] \rceil) - 2^{\lceil \text{Log2}[n] \rceil} + 1$$

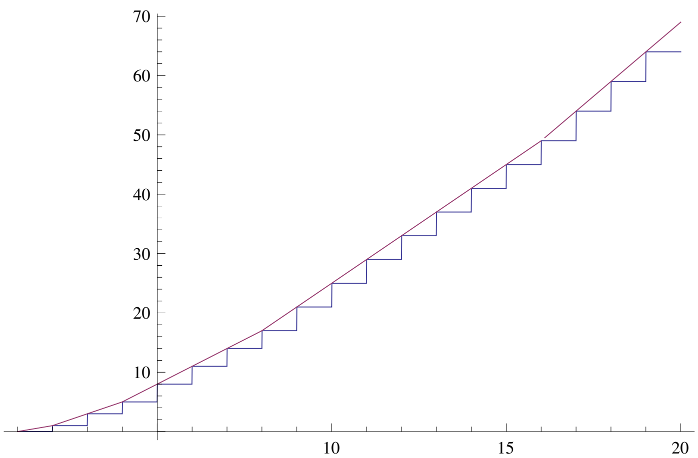
Also,

$$\sum_{i=1}^{n} \lceil \text{Log2}[i] \rceil = \sum_{i=1}^{n-1} \lfloor \text{Log2}[i] \rfloor + n - 1 =$$
$$= n (\text{Log2}[n] + \epsilon[n]) - 2 (n-1) + (n-1) =$$
$$= n (\text{Log2}[n] + \epsilon[n]) - n + 1$$

So,

$$\sum_{i=1}^{n} \lceil \text{Log2}[i] \rceil = n (\lceil \text{Log2}[n] \rceil) - 2^{\lceil \text{Log2}[n] \rceil} + 1$$

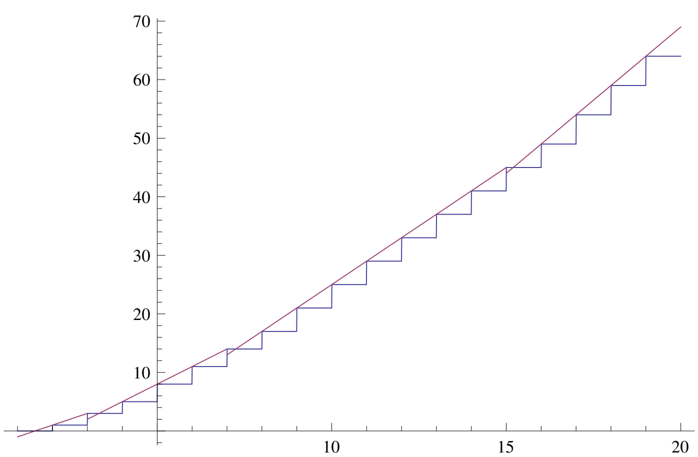
Plot$\left[\text{Tooltip}\left[\left\{\sum_{i=1}^{n} \lceil \text{Log2}[i] \rceil, n (\lceil \text{Log2}[n] \rceil) - 2^{\lceil \text{Log2}[n] \rceil} + 1\right\}\right], \{n, 1, 20\}\right]$



Also,

$$\sum_{i=1}^{n} \lceil \text{Log2}[i] \rceil = n (\lceil \text{Log2}[n+1] \rceil) - 2^{\lceil \text{Log2}[n+1] \rceil} + 1$$

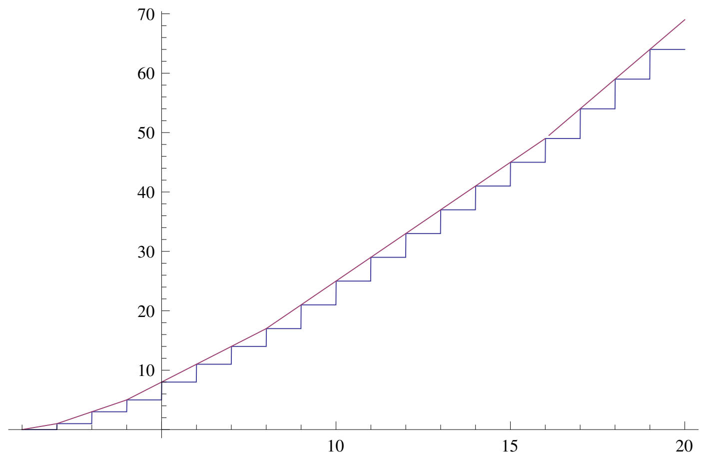
Plot$\left[\text{Tooltip}\left[\left\{\sum_{i=1}^{n} \lceil \text{Log2}[i] \rceil, n (\lceil \text{Log2}[n+1] \rceil) - 2^{\lceil \text{Log2}[n+1] \rceil} + 1\right\}\right], \{n, 1, 20\}\right]$



And

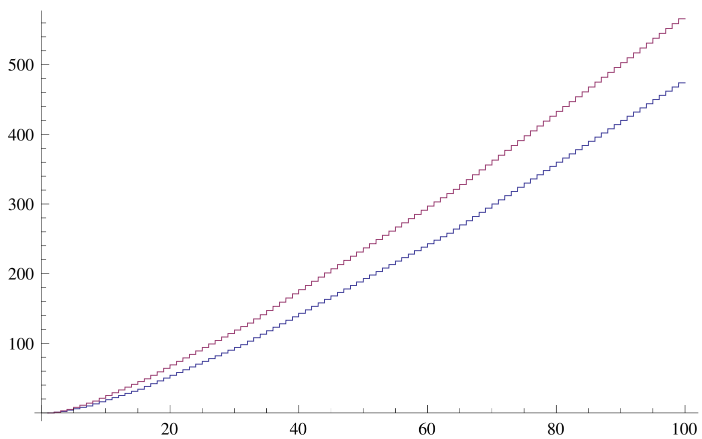
$\sum_{i=1}^{n} \lceil \text{Log2}[i] \rceil = n\,(\text{Log2}[n] + \epsilon[n]) - n + 1$

```
Plot[Tooltip[{∑_{i=1}^{n} ⌈Log2[i]⌉, n (Log2[n] + ϵ[n]) - n + 1}], {n, 1, 20}]
```



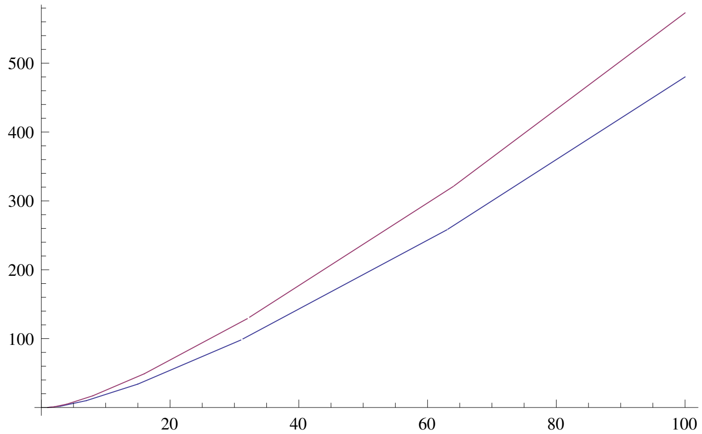
Sum of floors of logs and sum of ceilings of logs plotted together :

```
Plot[Tooltip[{∑_{i=1}^{n} ⌊Log2[i]⌋, ∑_{i=1}^{n} ⌈Log2[i]⌉}], {n, 1, 100}, PlotPoints → 200]
```



Their closed - form formulas plotted together :

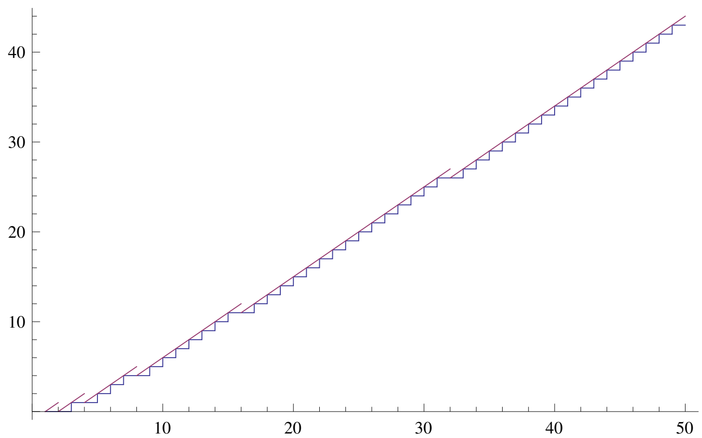
```
Plot[Tooltip[{(n + 1) ⌊Log2[n + 1]⌋ - 2^(⌊Log2[n+1]⌋+1) + 2, n (⌈Log2[n]⌉) - 2^⌈Log2[n]⌉ + 1}],
 {n, 1, 100}, PlotPoints → 200]
```



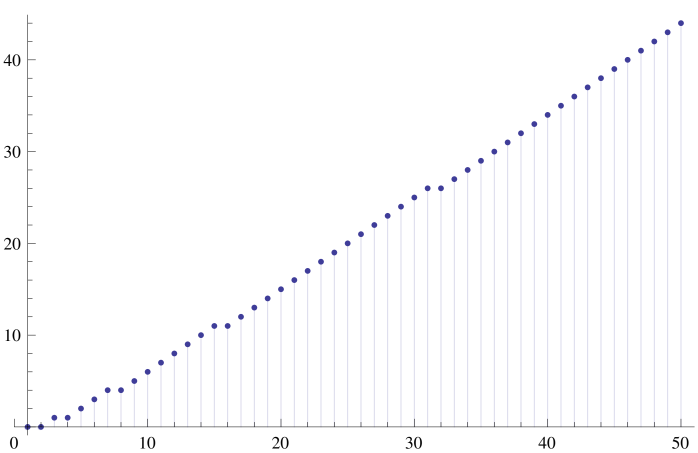
It follows from the above that

$$\sum_{i=1}^{n} (\lceil \text{Log2}[i] \rceil - \lfloor \text{Log2}[i] \rfloor) = n - \lceil \text{Log2}[n+1] \rceil = n - \lfloor \text{Log2}[n] \rfloor - 1$$

```
Plot[Tooltip[{∑_{i=1}^{n} (⌈Log2[i]⌉ - ⌊Log2[i]⌋), n - ⌊Log2[n]⌋ - 1}], {n, 1, 50}, PlotPoints → 200]
```

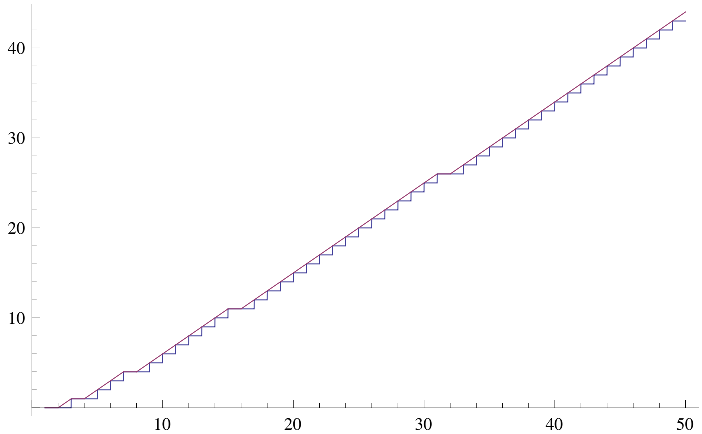



```
DiscretePlot[n - ⌊Log2[n]⌋ - 1, {n, 1, 50}]
```

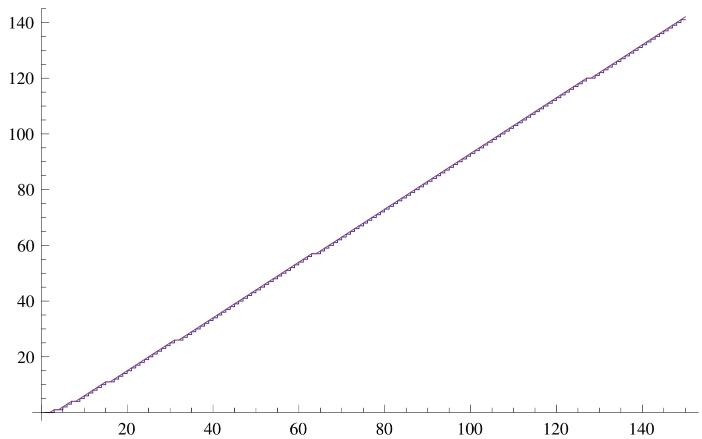


The same using continuous functions :

```
Plot[Tooltip[{∑_{i=1}^{n} (⌈Log2[i]⌉ - ⌊Log2[i]⌋), -((n + 1) ⌊Log2[n + 1]⌋ - 2^(⌊Log2[n+1]⌋+1) + 2)
    + (n (⌈Log2[n]⌉) - 2^⌈Log2[n]⌉ + 1)}], {n, 1, 50}, PlotPoints → 200]
```




```
Plot[Tooltip[{∑_{i=1}^{n} (⌈Log2[i]⌉ - ⌊Log2[i]⌋), -((n + 1) ⌊Log2[n + 1]⌋ - 2^(⌊Log2[n+1]⌋+1) + 2)
    + (n (⌈Log2[n]⌉) - 2^⌈Log2[n]⌉ + 1)}], {n, 1, 150}, PlotPoints → 400]
```




```
Plot[Tooltip[{∑_{i=1}^{n} (⌈Log2[i]⌉ - ⌊Log2[i]⌋), n - Log2[n] - 1}], {n, 1, 150}, PlotPoints → 300]
```