CSC 501/401 Analysis of Algorithms Spring '14

Dr. Marek A. Suchenek ©

CSUDH

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Formal definitions of Big-Oh and Big-Theta

Running times

Let $f : N \to R^+$ and $g : N \to R^+$, that is,

- f and g are functions (one may think of them as hypothetical running times of some programs)
- that take an integer n (the size of input) as an argument
- and return a positive real (a running time for an input of that size) as values $f(n)$ or $g(n)$, respectively.

Formal definitions of Big-Oh and Big-Theta

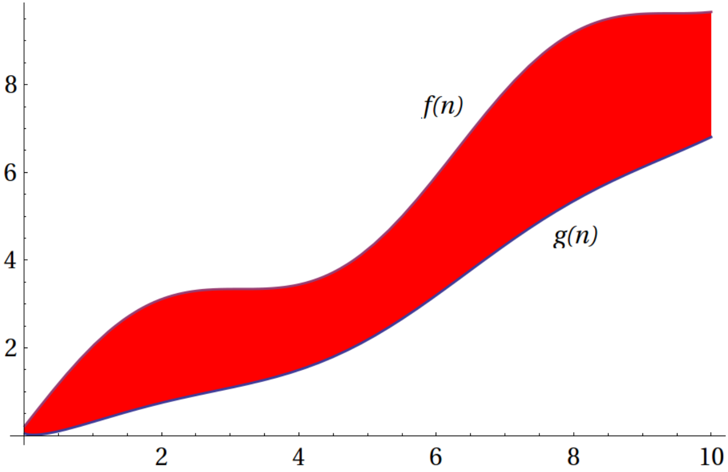

Figure: An example of f and g.

Definition

$$f \in O(g) \equiv \exists k \in R^{+}, \exists n_0 \in N, \forall n \geq n_0, f(n) \leq k \times g(n)$$

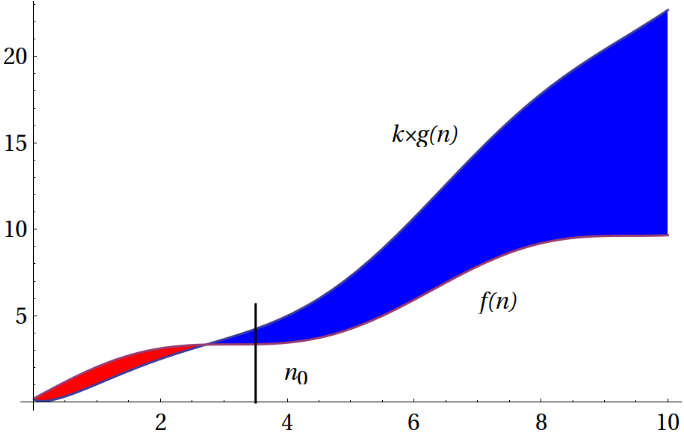

Figure: An example of k and n_0 that shows $f \in O(g)$.

Formal definitions of Big-Oh and Big-Theta

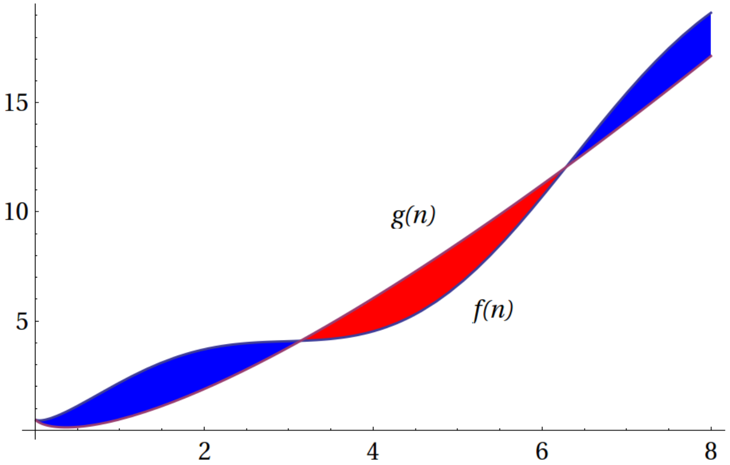

Figure: Another example of f and g.

Definition

$$f \in \Theta(g) \equiv$$

$$\equiv \exists k_1, k_2 \in R^+, \exists n_0 \in N, \forall n \geq n_0, k_1 \times g(n) \leq f(n) \leq k_2 \times g(n)$$

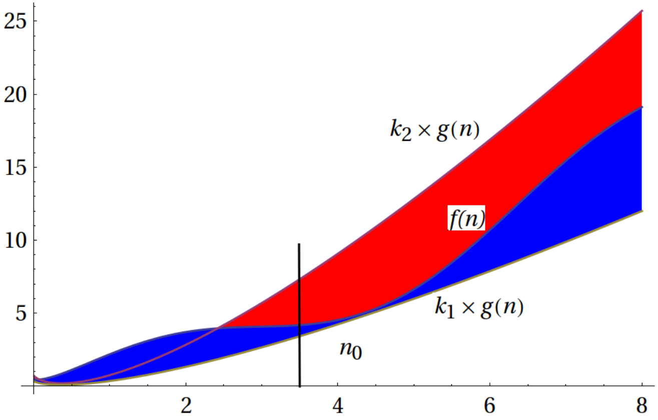

Figure: An example of k_1, k_2, and n_0 that show $f \in \Theta(g)$.

Properties of Big-Oh and Big-Theta

Fact

$$f \in \Theta(g) \equiv f \in O(g) \wedge g \in O(f)$$

Fact

If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ *exists then*

$$f \in O(g) \equiv \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty.$$

Fact

If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ *exists then*

$$f \in \Theta(g) \equiv 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty.$$

De l'Hôpital rule

Theorem

Assume that f and g are differentiable functions, $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$ or ∞, and that the limit $\lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ exists. Then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

Properties of Big-Oh and Big-Theta

Example

We will show that

$$n \log n \in O(n^2).$$

It suffices to show that

$$\lim_{n\to\infty} \frac{n \log n}{n^2} < \infty.$$

Indeed,

$$\lim_{n\to\infty} \frac{n \log n}{n^2} = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\log' n}{n'} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} =$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0 < \infty.$$

Properties of Big-Oh and Big-Theta

The following two facts are mandatory for graduate students and optional for undergraduete students.

Fact

$$f \in O(g) \equiv \overline{\lim_{n\to\infty}} \frac{f(n)}{g(n)} < \infty.$$

Fact

$$f \in \Theta(g) \equiv 0 < \underline{\lim_{n\to\infty}} \frac{f(n)}{g(n)} \wedge \overline{\lim_{n\to\infty}} \frac{f(n)}{g(n)} < \infty.$$

Formal definition of little-oh

Definition

$$f \in o(g) \equiv \forall K \in R^{+}, \exists n_0 \in N, \forall n \geq n_0, f(n) \leq K \times g(n)$$

Fact

$$f \in o(g) \equiv \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Formal definition of Big-Omega

Definition

$$f \in \Omega(g) \equiv g \in O(f)$$

Fact

$$f \in \Omega(g) \equiv \lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$$

END